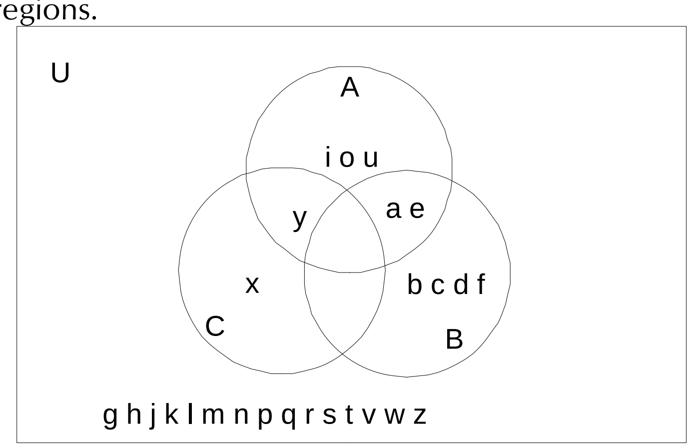
MC 215 MATHEMATICAL REASONING AND DISCRETE STRUCTURES, FALL 2008
Solutions to Hand-in Homework #1 – 9/17/08
32 points total (+ 2% extra credit for word-processing)

1. (12 pts) Consider the following subsets of the universal set U = {a, b, c, ..., x, y, z}, i.e., the 26 lower-case letters of the alphabet:
   A = {a, e, i, o, u, y}   B = {a, b, c, d, e, f}   C = {x, y}

   a. List all sets that are subsets of both A and B.

   This equals $\mathcal{P}(A \cap B) = \mathcal{P}(\{a, e\}) = \{\emptyset, \{a\}, \{e\}, \{a, e\}\}$.

   b. Fill in a Venn diagram like the one below, with all 26 letters shown in their appropriate regions.



   c. List all the elements of A x C.

   A x C = {(a, x), (e, x), (i, x), (o, x), (u, x), (y, x), (a, y), (e, y), (i, y), (o, y), (u, y), (y, y)}

   d. List all the elements of $\mathcal{P}(C)$ and, without listing them, say how many elements are in $\mathcal{P}(\mathcal{P}(C))$.

   $\mathcal{P}(C) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Since $|\mathcal{P}(C)| = 4$, $|\mathcal{P}(\mathcal{P}(C))| = 2^4 = 16$.

2. (4 pts) We determined in class that if $|S| = n$, then $|\mathcal{P}(S)| = 2^n$, and I said that if you keep taking power sets of power sets, the cardinalities get very big, very fast. Using a calculator that can perform the function x^y, and starting with $S = \emptyset$, compute the cardinalities of S, $\mathcal{P}(S)$, $\mathcal{P}(\mathcal{P}(S))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$, until your calculator can't display an answer. For example, your first three numbers will be $0 = |S|$, $1 = 2^0 = |\mathcal{P}(S)|$, and $2 = 2^1 = |\mathcal{P}(\mathcal{P}(S))|$. In other words, each time you get an answer x, your next answer is 2^x. If you do this correctly, you should find that your calculator can't handle many more such

calculations. You should write down (a) all the answers you get, (b) what was the largest number of "$\mathcal{P}$" operations your calculator could handle, and (c) the brand and model of calculator you used.

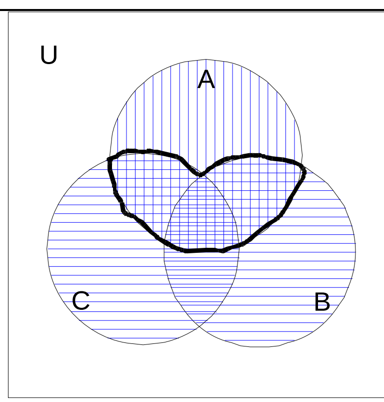
Using the calculator program on my Palm TX, I was able to handle 5 "$\mathcal{P}$" operations:

Set S:	$\emptyset$	$\mathcal{P}(\emptyset)$	$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}^2(\emptyset)$	$\mathcal{P}^3(\emptyset)$	$\mathcal{P}^4(\emptyset)$	$\mathcal{P}^5(\emptyset)$	$\mathcal{P}^6(\emptyset)$
|S|:	0	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^4 = 16$	$2^{16} = 65{,}536$	$2^{65{,}536}$ gives error

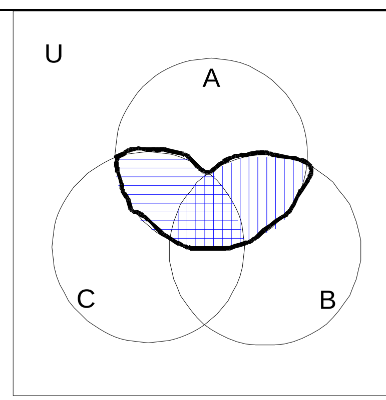
Note: Mathematica, with its default settings, *can* display the digits of $2^{65{,}536}$, and it's quite something to see!

3. (6 pts) Starting with two unshaded Venn diagrams like the one shown above, verify the first distributive law, A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C), by shading one Venn diagram to represent the left-hand expression, and the other Venn diagram to represent the right-hand expression, demonstrating that they represent the same regions.

In the left-hand Venn diagram below, A has been cross-hatched vertically, and B ∪ C has been cross-hatched horizontally. The intersection, A ∩ (B ∪ C), has both types of cross-hatching, and its border is drawn in a heavy black line. In the right-hand Venn diagram, (A ∩ B) has been cross-hatched vertically, and (A ∩ C) has been cross-hatched horizontally. The union, (A ∩ B) ∪ (A ∩ C), has either type of cross-hatching, and its border is also drawn in a heavy black line. The two regions are the same, demonstrating that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).



A ∩ (B ∪ C)



(A ∩ B) ∪ (A ∩ C)

4. (6 pts) As discussed in class, the mathematical "or" is *inclusive*, meaning that if both p and q are true, then so is p ∨ q. In standard English usage, the word "or" is often intended to be *exclusive*; for example, if I say, "I'm going to order in pizza tonight *or* I'm going to go out to PJ's Barbeque," it's understood that I won't do both. The "exclusive or" operator, denoted by the symbol XOR or ⊕, defines the proposition *p* XOR *q* to be true if *p* or *q* is true, but *false* if both or neither are true. In other words, *p* XOR *q* has the following truth table:

p	q	p XOR q
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Write a compound expression in *p* and *q*, which may use any of the operators ∧, ∨, or ¬, but *no others,* and which is equivalent to *p* XOR *q*. Prove this by constructing a truth table for your expression that shows the intermediate values, and whose final values agree with those shown above.

There was a surprising variety of correct expressions, including the following:
1) (p ∨ q) ∧ ¬(p ∧ q)
2) (p ∧ ¬q) ∨ (¬p ∧ q)
3) (p ∨ q) ∧ (¬p ∨ ¬q)
4) ¬(p ∧ q) ∧ ¬(¬p ∧ ¬q)
5) ¬((p ∧ q) ∨ (¬(p ∨ q)))
#2 was probably the most popular, so I'll do the table for that:

p	q	(p ∧ ¬q)	(¬p ∧ q)	(p ∧ ¬q) ∨ (¬p ∧ q)
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**F**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**F**

Since these truth values agree with those for p XOR q, we have shown that they are equivalent.

5. (4 pts) An operation is called *associative* if, when you perform it multiple times, it doesn't matter how you parenthesize. For example, addition is associative, because (x + y) + z = x + (y + z); however subtraction is *not* associative: (x – y) – z does not always equal x – (y – z). Determine whether the implication operator is associative. In other words, is the expression (p ⟶ q) ⟶ r equivalent to the expression p ⟶ (q ⟶ r)? Justify your answer by using truth tables or some other convincing argument.

The two expressions are not equivalent; hence the implication operation is not associative. To prove this, we could give a complete truth table, with 8 rows of values for p, q, and r, demonstrating that the two expressions have different sets of truth values.. Alternatively, we can give just one triple of values for which they do not agree. For example, if p, q, and r are all *false*, then:

(p ⟶ q) ⟶ r ≡ (F ⟶ F) ⟶ F ≡ T ⟶ F ≡ F, while

p ⟶ (q ⟶ r) ≡ F ⟶ (F ⟶ F) ≡ F ⟶ T ≡ T,

thus the two expressions are not equivalent. (Note: They also disagree when p = F, q = T, and r = F.)