

Solutions to Hand-in Homework #3 – 44 points total

Reminder: Proofs will be graded primarily on correctness, but also on presentation, including logical flow, reasonable justification of statements, and mechanics such as grammar and spelling.

1. For each $n \geq 2$, consider the sum $S(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n}$.

By examining $S(n)$ for small values of n , conjecture a simple formula for $S(n)$, and then prove it's true by induction.

Solution to #1 (8 pts): We prove by induction on $n \geq 2$ that $S(n) = \frac{n-1}{n}$. (Some people said

$S(n) = 1 - \frac{1}{n}$, which is equivalent.

Base case: When $n = 2$, $S(n) = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{n-1}{n}$.

Inductive case: We assume for some $n \geq 2$ that $S(n) = \frac{n-1}{n}$, and we prove that

$$S(n+1) = \frac{n}{n+1}.$$

By definition, $S(n+1) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n-1) \cdot n} + \frac{1}{n(n+1)} = S(n) + \frac{1}{n(n+1)}$. Thus the

inductive assumption implies that $S(n+1) = S(n) + \frac{1}{n(n+1)} = \frac{n-1}{n} + \frac{1}{n(n+1)}$. Getting a

common denominator and simplifying yields

$$S(n+1) = \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1},$$

which proves the inductive case. By the Principle of Mathematical Induction, we conclude that the claim is true for all $n \geq 2$.

2. Use induction to prove that any triangulation of a polygon with n sides, $n \geq 3$, has exactly $n - 2$ triangular regions. *Before doing this problem, be sure to read the material on polygons, diagonals, and triangulations, which appears after the last problem.*

Solution to #2 (8 pts): We prove this by induction on $n \geq 3$. In the base case, the polygon is itself a triangle with no diagonals, hence a triangulation with $1 = 3 - 2$ triangular regions. For the inductive case, let $n \geq 3$, and assume that any triangulated polygon with k sides, $3 \leq k \leq n$, has exactly $k - 2$ triangular regions (we are using *strong* induction). Suppose now that we have a triangulated polygon P with $n + 1$ sides. Since $n + 1 > n \geq 3$, the lemma given implies there is at least one diagonal d . We break P along d into two smaller polygons, P_1 and P_2 , with k_1 and k_2 sides, respectively, and each inheriting a triangulation from the triangulation of P . Since d is an edge of both P_1 and P_2 , and each edge of P is an edge of P_1 or P_2 , but not both, we have that $k_1 + k_2 = 2 + n + 1 = n + 3$. Now d is a diagonal, so the path from one endpoint of d to its other endpoint must have length at least 2 in both P_1 and P_2 . If we count the number of edges in P_1 , this means that at least two of P 's original edges are *not* in P_1 , because they are in P_2 , so when we add the one new edge d , P_1 still has at least one less edge than P . Switching the roles of P_1 and P_2 , this is also true of P_2 . Hence our (strong) inductive assumption applies to P_1 and P_2 , and it implies that the triangulations of P_1 and P_2 have $k_1 - 2$ and $k_2 - 2$ triangles, respectively. The sum of these two numbers equals the number of triangles in the triangulation P , hence the number of triangles in the triangulation of P equals $k_1 - 2 + k_2 - 2 = n + 1 - 2 = n - 1$, which proves the inductive case. By the Principle of Mathematical Induction, we conclude the claim is true for all $n \geq 3$.

(#4a from the **Countability Handout** on our website). Given that \mathbf{Q}^+ , the set of positive rational numbers, is countable, prove that \mathbf{Q} , the set of *all* rational numbers is countable. **Hint:** Since \mathbf{Q}^+ is countable, its elements can be listed sequentially: $\mathbf{Q}^+ = \{q_1, q_2, q_3, \dots\}$. Use this fact to construct a sequential listing of the elements of \mathbf{Q} .

Solution to #3 (5 pts):

Since \mathbf{Q}^+ is countable we can write it sequentially as $\mathbf{Q}^+ = \{q_1, q_2, q_3, \dots\}$. Thus we can write \mathbf{Q} sequentially as $\mathbf{Q} = \{0, q_1, -q_1, q_2, -q_2, \dots\}$, hence \mathbf{Q} is countable.

3. (#5 from the Countability Handout) Parts (a) and (b) of this exercise together comprise a proof of Theorem 9.
- a. Let S be any nonempty set. Construct a one-to-one function from S to $\mathcal{P}(S)$, which proves that $|S| \leq |\mathcal{P}(S)|$.
 - b. Let S be any nonempty set. Prove that there is no bijection from S to $\mathcal{P}(S)$, by completing the following proof by contradiction: Suppose there is a bijection $f: S \rightarrow \mathcal{P}(S)$, i.e., for each $s \in S$, $f(s)$ is a subset of S . Consider the following set C :

$$\mathbf{C} = \{s \in S \mid s \notin f(s)\}$$

If f is a bijection, then there must be some element $c \in S$ such that $f(c) = C$. Now either $c \in C$ or $c \notin C$. Prove that in either case the definition of C leads to a contradiction, so that we conclude that f could not have been a bijection.

Solution to #4, part a (5 pts): Define the function $f: S \rightarrow \mathcal{P}(S)$, by $f(s) = \{s\}$. Then for any $s, t, \in S$, if $f(s) = f(t)$, then $\{s\} = \{t\}$, which implies that $s = t$. Hence f is one-to-one.

Solution to #4, part b (8 pts): We prove the claim by contradiction, so suppose $f: S \rightarrow \mathcal{P}(S)$ is a bijection. Then, in particular, f is onto. Define the set $C = \{s \in S \mid s \notin f(s)\}$. This says, for any element $s \in S$,

- (1) If $s \in C$, then $s \notin f(s)$, and
- (2) If $s \notin C$, then $s \in f(s)$.

Since $C \in \mathcal{P}(S)$ and f is onto, there is at least one $x \in S$ such that $f(x) = C$. Now either $x \in C$ or $x \notin C$; we find a contradiction in either case.

First suppose $x \in C$. From (1) we have that $x \notin f(x)$, but $f(x) = C$, so this says $x \notin C$, the negation of what we assumed, hence a contradiction. If we assume instead that $x \notin C$, then (2) implies that $x \in f(x)$, and so the definition of C implies that $x \in C$. Again this negates our assumption that $x \notin C$, so we have arrived at a contradiction in either case. We conclude that there can be no x with $f(x) = C$, and thus f is not onto. Therefore f is not a bijection, proving that no function from S to $\mathcal{P}(S)$ is a bijection.

4. (p. 147 #131, rephrased slightly) A *palindrome over X* is a string α for which $\alpha = \alpha^R$ (i.e., a string that reads the same forward and backward). Let $X = \{a, b\}$. An example of a palindrome over X is *bbbaabb*. For the function $f: X^* \rightarrow X^*$ defined by $f(\alpha) = \alpha\alpha^R$, determine whether (a) f is one-to-one, and (b) whether f is onto. Prove your answers.

Solution to #5a (5 pts). First we note that if a string α has length n , then α^R also has length n , and thus $f(\alpha) = \alpha\alpha^R$ has length $2n$, with α equal to the substring of $f(\alpha)$ comprised of its first n characters. Suppose now that $f(\alpha) = f(\beta)$, and call this string s . Since $s = f(\alpha)$, the above reasoning says s has length $2n$, and if $s = s_1s_2\dots s_{2n}$, then $\alpha = s_1s_2\dots s_n$. But by the same reasoning, if $|\beta| = m$, then $s = f(\beta)$ implies that s has length $2m$, and $\beta = s_1s_2\dots s_m$. But then $|s| = 2n = 2m$ implies that $m = n$, and $\alpha = s_1s_2\dots s_n = s_1s_2\dots s_m = \beta$. Hence f is one-to-one.

Solution to #5b (5 pts). Since I accidentally made the codomain all of X^* , rather than the set of palindromes, this part is trivial. Any string that is not a palindrome is not in the range of f , e.g. the string *aabbbb*. But the more interesting observation is that if we define f as a function from X^* to \mathcal{P} = the set of palindromic strings over X^* (as it was defined in the text), it is still not onto. We showed in part a that for any x , $f(x)$ has even length, but palindromes can have odd length, e.g., *a* or *bbabb*. None of these palindromic strings are in the range of f .