<u>MC 215 MATHEMATICAL REASONING AND DISCRETE STRUCTURES – 9/26/08</u>

Equivalence of Weak and Strong Induction

The goal here is to prove that the the Principles of Weak and Strong Induction are logically equivalent. Since their conditions are so similar, the proof can be confusing. To try to clarify things, I've called Weak Induction Theorem 1, and Strong Induction Theorem 2. I've called their base cases 1w and 1s, and their inductive cases 2w and 2s. There are two proofs: one to show that Theorem 1 implies Theorem 2, and the other to show that Theorem 2 implies Theorem 1.

Theorem 1 (Weak Induction). Suppose P(n) is a statement for each integer $n \geq 1$, and suppose the following two conditions hold:

(1w) P(1) is true;

(2w) For all $n \geq 1$, if P(n) is true, then P(n+1) is true.

Then P(n) is true for all $n \geq 1$.

Theorem 2 (Strong Induction). Suppose P(n) is a statement for each integer $n \geq 1$, and suppose the following two conditions hold:

(1s) P(1) is true;

(2s) For all $n \geq 1$, if $P(1) \wedge P(2) \wedge \ldots \wedge P(n)$ is true, then P(n+1) is true.

Then P(n) is true for all $n \geq 1$.

I'll do the easier proof first.

Proof that Theorem 2 implies Theorem 1. We assume that Theorem 2 is true, i.e., whenever conditions (1s) and (2s) hold for a statement P(n), then P(n) is true for all $n \geq 1$. We must prove that Theorem 1 is true, i.e., we must prove that if conditions (1w) and (2w) hold for a statement P(n), then P(n) is true for all $n \geq 1$.

Suppose that P(n) is statement that satisfies conditions (1w) and (2w) of Theorem 1. We want to show that P(n) is true for all $n \geq 1$. To do this we show that P(n) satisfies conditions (1s) and (2s) of Theorem 2, and so by that Theorem, which we've assumed is true, P(n) is true for all $n \geq 1$.

Note that condition (1w) = condition (1s), so assuming that (1w) holds implies that (1s) holds. We assume that condition (2w) holds, so for any $n \geq 1$, if P(n) is true, then so is P(n+1). For condition (2s) we assume that $P(1) \wedge P(2) \wedge \ldots \wedge P(n)$ is true. But this implies that P(n) is true. so by our assumption of condition (2w), it follows that P(n+1) is true. Thus we've shown that P(n) satisfies condition (2w). Since Theorem 2 is true, it follows that P(n) is true for all $n \geq 1$. This proves that Theorem 1 is true, since we began with the assumptions (1w) and (2w). ∎

Proof that Theorem 1 implies Theorem 2. We assume that Theorem 1 is true, i.e., whenever conditions (1w) and (2w) hold for a statement P(n), then P(n) is true for all $n \geq 1$. We must prove that Theorem 2 is true, i.e., we must prove that if conditions (1s) and (2s) hold for a statement P(n), then P(n) is true for all $n \geq 1$.

The outline of this proof is the following:

- We assume that conditions (1s) and (2s) of Theorem 2 hold for a statement P(n); to show that Theorem 2 is true, we must show that P(n) is true for all n ≥ 1.
- We use assumptions (1s) and (2s) to apply Theorem 1 to a *different* statement Q(n), and conclude that Q(n) is true for all n ≥ 1.
- We finish the proof by noting that if Q(n) is true for all n ≥ 1, then so is P(n), which is what we wanted to prove.

So, suppose we have a statement P(n) for which conditions (1s) and (2s) are true. Define a new statement **Q(n) = P(1) ^ P(2) ^ … ^ P(n)**. We use Theorem 1 (which we're assuming is true) to show that Q(n) is true for all n. For condition (1w), Q(1) must be true, but Q(1) = P(1) which is true by our assumption that condition (1s) holds for P(n). For condition (2w), we must show that if n ≥ 1, and Q(n) is true, then Q(n+1) is true. Now Q(n) = P(1) ^ P(2) ^ … ^ P(n), and we are assuming that condition (2s) holds for P(n), which says that if P(1) ^ P(2) ^ … ^ P(n) is true, then so is P(n+1). Thus we have that P(1) ^ P(2) ^ … ^ P(n) is true *and* P(n+1) is true, and it follows that P(1) ^ P(2) ^ … ^ P(n) ^ P(n+1) = Q(n+1) is true. This shows that Q(n) satisfies condition (1w). We now apply Theorem 1 to conclude that Q(n) is true for all n ≥ 1. But if Q(n) = P(1) ^ P(2) ^ … ^ P(n) is true, then certainly P(n) is true, so this implies that P(n) is true for all n ≥ 1.

Thus, starting with the assumptions that Theorem 1 is true, and that condition (1s) and (2s) hold for P(n), we've shown that P(n+1) is true for all n ≥ 1. This proves that Theorem 2 is true. ∎