

MC 215 -- MATHEMATICAL REASONING AND DISCRETE STRUCTURES --
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The Erdős–Szekerer Theorem

In class we looked at the sequence $\{86, 12, 63, 82, 78, 43, 19, 75, 56, 81\}$. The longest increasing subsequence we could find had length 4: $\{12, 19, 56, 81\}$. The longest decreasing subsequence had length 5: $\{86, 82, 78, 75, 56\}$. Notice that the original sequence had 10 elements, and the two subsequences had 4 elements and 5 elements, respectively. How long would the original sequence have to be to *guarantee* it has a subsequence of at least 4 elements that's either increasing or decreasing? This is an example of a question answered by the Erdős–Szekerer Theorem, which says how long a sequence must be to guarantee a subsequence of length n that is either monotonically increasing or monotonically decreasing, for any $n \geq 1$. Notice by the way that if all the elements in the sequence are different, then nondecreasing is the same as increasing, and nonincreasing is the same as decreasing.

- For example, if $n = 1$, then any sequence with just 1 element is itself an increasing (and decreasing) subsequence with one element.
- If $n = 2$, again any sequence with just 2 elements works, since either $a_1 \leq a_2$, making the sequence monotonically increasing, or $a_1 \geq a_2$, making it monotonically decreasing.
- If $n = 3$, then not every sequence with 3 elements works. For instance, the sequence $\{1, 3, 2\}$ has neither an increasing nor decreasing subsequence of length 3. Can you come up with a sequence of length 4 that has neither a nonincreasing nor nondecreasing subsequence of length 3? For $n = 3$, the Erdős–Szekerer Theorem says that length 4 the longest sequence that fails: every sequence with length 5 or more has either a nondecreasing or nonincreasing subsequence of length 3.

Before stating the theorem, I'll point out that it's phrased in terms of wanting to guarantee a subsequence of length $n + 1$, instead of n , since that makes the formulas look nicer. So if we want a subsequence of length 3, then we let $n = 2$, so that $n + 1 = 3$. Here's the theorem:

The Erdős–Szekerer Theorem. Suppose $n \geq 0$ is an integer. Then any sequence of real numbers with length at least $n^2 + 1$ has either a nondecreasing subsequence of length $n + 1$ or a nonincreasing subsequence of length $n + 1$.

So for $3 = 2 + 1$, the theorem says that any sequence of length $2^2 + 1 = 5$ has such a subsequence. To guarantee a nonincreasing or nondecreasing subsequence of length 4, we let $n = 3$, since $4 = 3 + 1$, and the theorem says that any sequence with at least $3^2 + 1 = 10$ elements has such a subsequence of length 4.

Proof. Let $S = \{s_i\}_{i=1}^{n^2+1}$ be any sequence of real numbers with $n^2 + 1$ elements. We look at two cases: (1) S has a nondecreasing subsequence of length $n + 1$, or (2) S does *not* have a nondecreasing subsequence of length $n + 1$. Clearly S satisfies either Case 1 or Case 2. If S satisfies Case 1, we're done. Otherwise S satisfies Case (2) and in this case, we show that S must have a nonincreasing subsequence of length $n + 1$.

(Here comes the clever – and tricky -- part!) Assuming that S has no nondecreasing subsequence of length $n + 1$, look at each element s_i of S and determine what is the *longest* nondecreasing subsequence that **starts with s_i** . For each i , $i = 1, \dots, n^2 + 1$, the answer may be different, but in every case it has to be at most n , since we are assuming that *no* nondecreasing subsequence of S has length $n + 1$. Now, for each possible length k from 1 to n , we count how many elements of S begin a longest nondecreasing subsequence of length k ; call this number L_k . Thus for each k , $k = 1, \dots, n$, L_k is the number of elements of S that begin a longest nondecreasing subsequence of length k .

Now S has $n^2 + 1$ elements. If every $L_k \leq n$, then $\sum_{k=1}^n L_k \leq \sum_{k=1}^n n = n^2$. But the sum of the

L_k 's equals the number of elements in S . So at least one L_k must be $\geq n + 1$. This means that there is a subsequence of length $n + 1$, such that for every number s_i in the subsequence, the longest nondecreasing sequence starting with s_i has length exactly k . So call the elements of this subsequence, in the order that they appear in S , a_1, a_2, \dots, a_{n+1} . We show that $a_1 > a_2 > \dots > a_{n+1}$, i.e., it is a *decreasing* sequence of length $n + 1$.

We prove $a_1 > a_2 > \dots > a_{n+1}$ by contradiction. Suppose $a_1 \leq a_2$. Then any nondecreasing sequence beginning with a_2 can be made one element longer by putting a_1 before a_2 . But this contradicts the assumption that the longest nondecreasing subsequence starting with either a_1 or a_2 has the same length k . Similarly, it's impossible that $a_2 \leq a_3$, that $a_3 \leq a_4$, ..., and that $a_n \leq a_{n+1}$. Therefore it must be the case that $a_1 > a_2 > \dots > a_{n+1}$.

Thus, either $S = \{s_i\}_{i=1}^{n^2+1}$ has a nondecreasing subsequence of length $n + 1$, or it has a nonincreasing (in fact, decreasing) subsequence of length $n + 1$. ■

The E-S Theorem is actually a “best possible” result, in the sense that if you have a sequence with fewer than $n^2 + 1$ elements, it need not have a nonincreasing or nondecreasing subsequence of length n . We can prove this by showing that we can always construct a sequence of length n^2 that has no nonincreasing or nondecreasing subsequence of length $n + 1$. Let's begin with an example, using $n + 1 = 4$, i.e., $n = 3$. The theorem says we can always do it if our original sequence has $n^2 + 1 = 10$ elements, but here's a sequence with $n^2 = 9$ elements that doesn't have either a nonincreasing or nondecreasing subsequence of length 4:

3, 6, 9, 2, 5, 8, 1, 4, 7

To see that there is no such subsequence, look at this arrangement of the sequence, in order from left to right, but in n rows:

		9			8			7
	6			5			4	
3			2			1		

If you look at any number, you see that all the higher numbers that come after it are in higher rows. So any *increasing* subsequence must go up at least one row, as it goes from left to right. There are only 3 rows, so the longest increasing subsequence has length 3. To see that the longest *decreasing* subsequence also has length at most 3, notice that the first three elements are increasing (3, 6, 9), as are the second three (2, 5, 8) and the third three (1, 4, 7). So the longest decreasing subsequence would have one from the first 3 (3, 6, 9), one from the second three (2, 5, 8), and one from the third three (1, 4, 7), hence the longest decreasing subsequence also has length 3.

This can be generalized to arbitrary n : The same basic argument can be used to show that the following sequence of length n^2 has no increasing or decreasing subsequence of length $n + 1$:

$n, 2n, \dots, n^2, n-1, 2n-1, \dots, n^2-1, n-2, 2n-2, \dots, n^2-2, \dots, \dots, 1, 1+n, \dots, 1+n^2-n$

			n^2				n^2-1	...				$1+n^2-n$
		
	$2n$				$2n-1$...		$1+n$		
n				$n-1$...	1			