

MC215: MATHEMATICAL REASONING
AND DISCRETE STRUCTURES

- **Monday, 9/22/08 & Wednesday, 9/24/08**
 - From last time:
 - Proof exercises
 - Today:
 - Existence and uniqueness
 - Proof by induction
- READING: 2.4 (omit 2.3)
- EXERCISES:
 - pp.86-87: 12, 14, 15, 47
 - pp. 102-105: 1, 4, 17, 29, 30

Monday-Wednesday, 9/22-24/08, Slide #1

Existence and Uniqueness

- The symbol ! has two meanings in math:
 - **Factorial:** $n!$ means "n factorial:"
 $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$
 - **Uniqueness:** $\exists ! x Q(x)$ means "there exists a *unique* x such that Q(x)."
- Proving existence and uniqueness:
 - **Existence:** Produce a **single specific value of x** that makes Q(x) true.
 - **Uniqueness:** Prove that, if x_1 and x_2 both make Q(x) true, then $x_1 = x_2$, i.e.,

$$Q(x_1) \wedge Q(x_2) \Rightarrow x_1 = x_2.$$

Monday-Wednesday, 9/22-24/08, Slide #2

Example

- **Theorem.** If a, b, and c are real numbers, and $a \neq 0$, then there exists a unique solution to the equation $ax + b = c$.
- **Proof:**
 - **Existence:**
 - **Uniqueness:**

Monday-Wednesday, 9/22-24/08, Slide #3

A puzzle

- I have a bunch of blocks numbered 1, 2, 3, 4, ..., and some have an X marked on the bottom. I know two things:
 - 1. If any block # k has an X on the bottom, then block # $k+1$ also has an X on it.
 - 2. Block #1 has an X on the bottom.
- Which blocks have an X on the bottom?



Monday-Wednesday, 9/22-24/08, Slide #4

Principle of Mathematical Induction

- Theorem (Principle of Mathematical Induction).** Suppose $S(n)$ is a statement for n a positive integer, and suppose that
 - $S(1)$ is true; and
 - For any $n \geq 1$, $S(n) \Rightarrow S(n+1)$.Then $S(n)$ is true for all $n \geq 1$.
- Proof by contradiction.**
 - Suppose all our assumptions are true, but $S(n)$ is *not* true for all $n \geq 1$.
 - Let k be the smallest value of n such that $S(k)$ is false.
 - $k \neq 1$ – why?
 - So $k > 1$. What's the contradiction?

Monday-Wednesday, 9/22-24/08, Slide #5

Template for using PMI in proofs.

- Theorem.** $S(n)$ is true for all $n \geq 1$.
- Proof by Mathematical Induction.**
 - Base case:** Prove $S(1)$ is true.
 - Inductive case:**
 - Assume $n \geq 1$ and $S(n)$ is true.
 - Prove $S(n+1)$ is true.
 - By the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 1$.** ■
- The assumption " $n \geq 1$ and $S(n)$ is true" is called the *inductive assumption*.

Monday-Wednesday, 9/22-24/08, Slide #6

Example 1

- **Theorem.** For all $n \geq 1$, $5^n - 1$ is divisible by 4.
- **Proof by Mathematical Induction.**
 - **Base case.** Prove the theorem is true when $n = 1$.
 - **Inductive case.**
 - **Assume:** $n \geq 1$, and $5^n - 1$ is divisible by 4.
 - **Prove:** $5^{n+1} - 1$ is divisible by 4.
 - **Conclude:** The theorem is true by the Principle of Mathematical Induction. ■

Monday-Wednesday, 9/22-24/08, Slide #7

Example 2

- **Theorem.** For all $n \geq 1$, $n < 2^n$.
- **Proof by Mathematical Induction.**
 - **Base case.** Prove the theorem is true when $n = 1$.
 - **Inductive case.**
 - **Assume:** $n \geq 1$, and $n < 2^n$.
 - **Prove:** $n+1 < 2^{n+1}$.
 - **Conclude:** The theorem is true by the Principle of Mathematical Induction. ■

Monday-Wednesday, 9/22-24/08, Slide #8

More examples: Summations

- The symbol Σ means "summation" in math.
- $\sum_{i=1}^n f(i)$ means "the summation of $f(i)$ from $i = 1$ to n ."
- $\sum_{i=1}^n f(i) = f(1) + f(2) + f(3) + \dots + f(n-1) + f(n)$
- $\sum_{i=1}^7 i = ??$ $\sum_{k=0}^4 \left(\frac{1}{2}\right)^k = ???$

Monday-Wednesday, 9/22-24/08, Slide #9

Summation Examples

□ Prove:

□ $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

□ $\sum_{k=0}^n \left(\frac{1}{2}\right)^k = 2 - \left(\frac{1}{2}\right)^n$

Monday-Wednesday, 9/22-24/08, Slide #10
