MC215: MATHEMATICAL REASONING AND DISCRETE STRUCTURES

- Monday, 9/15/08
  - From last time: Bad proofs
  - Disproof
  - Proof by cases
  - Indirect proof = proof of contrapositive
  - Proofs of equivalence = “if and only if” proofs
- READING: 2.2
- EXERCISES:
  - pp. 86-87: 3, 6, 21, 30, 31, 41

Disproof

- How do we disprove the universally quantified statement, "If P(x) then Q(x)"?
  - If it's true, then it's true for ***every x.***
  - If it's false, that means **there exists an x for which it's false.**

- **Find a counterexample:** An x for which P(x) is true, but Q(x) is false.

- **Example:** Disprove the statement, "If n is prime, then so is $n^2 - n + 11$."

Proof by cases

- Suppose we have a theorem of this form:

  If P(x) or Q(x) then R(x)

- This is equivalent to *two* theorems:
  - **If P(x) then R(x),** ***and***
  - **If Q(x) then R(x)**
- Direct proof by cases:
  - **Case 1. Assume P(x); prove R(x).**
  - **Case 2. Assume Q(x); prove R(x).**

Cases depending on parity

- Sometimes it helps to use cases, even when they're not explicit in the hypothesis:
  - A proof about integers can be different, depending on whether a variable is even or odd.
- **Theorem**. Suppose $n \in \mathbf{N}$ is a perfect square. Then either n or n-1 is a multiple of 4.
- **Proof.** Assume n is a perfect square. Then $n = m^2$ for some $m \in \mathbf{N}$.
  - **Case 1.** m is even.
  - **Case 2.** m is odd.

Cases depending on sign

- **Absolute value function:** For $x \in R$,
  - **$|x| = x$, if $x \geq 0$,**
    $-x$, if $x < 0$
- Proofs involving absolute value are frequently done in cases, depending on whether the variables involved are ≥ 0 or < 0.
- **Theorem**. $|xy| = |x|\,|y|$.
- **Proof?** Hint: Do 4 cases depending on whether x and y are ≥ 0 or < 0.

Indirect Proof

- **Indirect Proof** of an implication **P(x) ⇒ Q(x)** means proving its ***contrapositive.*** (It does *not* mean "proof by contradiction," as our text says.)
- Statement:
  - **If P(x) then Q(x).**
- Contrapositive:
  - **If ¬ Q(x) then ¬ P(x).**
- **Indirect Proof:**
  - **Assume Q(x) is false.**
  - **Prove P(x) is false.**

Example, proof strategy

- Give an indirect proof of: For n $\in$ **N**, if 3n + 2 is odd, then n is odd.
  - **Indirect Proof begins:** "We prove the contrapositive, so we assume that … . We must prove that … ."
- **Proof strategy: To prove "If P(x), then Q(x):"**
  - **FIRST:** Try out (at least think about) ***direct proof:***
    - Assume P(x) is *true*.
    - Prove Q(x) is *true*.
  - **SECOND:** If that seems hard, try out **indirect proof:**
    - Prove Q(x) is *false*.
    - Assume P(x) is *false*.

Equivalence, or "if and only if" proofs

- The statement, **$P(x) \Leftrightarrow Q(x)$**, is equivalent to the two statements, **$P(x) \Rightarrow Q(x)$ *and* $Q(x) \Rightarrow P(x)$**.
  - It's also written "**P(x) if and only Q(x)**," or for short, "**P(x) IFF Q(x)**."
- It's really *two* statements, which require *two separate proofs.*
  - **"Necessity (of Q) proof: $P(x) \Rightarrow Q(x)$"** Assume P(x), prove Q(x).
    - So Q(x) is a "necessary condition" for P(x)
  - **"Sufficiency (of Q) proof: $Q(x) \Rightarrow P(x)$"** Assume Q(x), prove P(x).
    - So Q(x) is a "sufficient condition" for P(x).

Example

- **Theorem.** An integer n is odd if and only if n^2 is odd.