# MC215: MATHEMATICAL REASONING AND DISCRETE STRUCTURES

- **Monday, 9/22/08 & Wednesday, 9/24/08**
  - From last time:
    - Proof exercises
  - Today:
    - Existence and uniqueness
    - Proof by induction
- READING: 2.4 (omit 2.3)
- EXERCISES:
  - pp.86-87: 12, 14, 15, 47
  - pp. 102-105: 1, 4, 17, 29, 30

# Existence and Uniqueness

- The symbol ! has two meanings in math:
  - **Factorial**: n! means "n factorial:" n! = n (n-1)(n-2) ... 3 2 1
  - **Uniqueness**: ∃ ! x Q(x) means "there exists a *unique* x such that Q(x)."
- Proving existence and uniqueness:
  - **Existence**: Produce a **single specific value of x** that makes Q(x) true.
  - **Uniqueness**: Prove that, if $x_1$ and $x_2$ both make Q(x) true, then $x_1 = x_2$, i.e.,

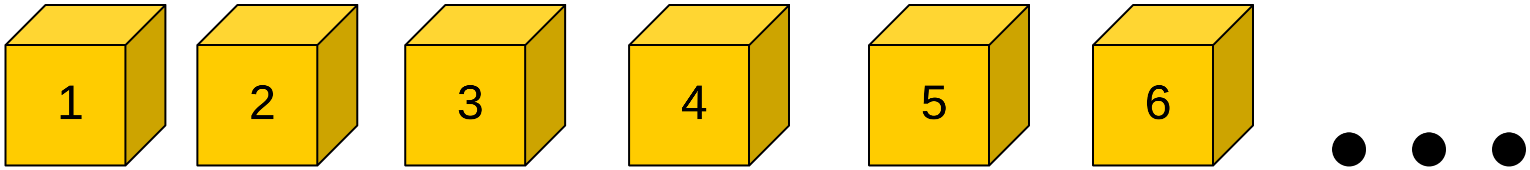
$$\mathbf{Q(x_1) \land Q(x_2) \Rightarrow x_1 = x_2.}$$

# Example

- **Theorem**. If a, b, and c are real numbers, and $a \neq 0$, then there exists a unique solution to the equation $a x + b = c$.
- **Proof**:
  - **Existence**:
  - **Uniqueness**:

# A puzzle

- I have a bunch of blocks numbered 1, 2, 3, 4, ..., and some have an X marked on the bottom. I know two things:
  - 1. If any block #k has an X on the bottom, then block #k+1 also has an X on it.
  - 2. Block #1 has an X on the bottom.
- Which blocks have an X on the bottom?



# Principle of Mathematical Induction

- **Theorem (Principle of Mathematical Induction).** Suppose S(n) is a statement for n a positive integer, and suppose that
  - S(1) is true; and
  - For any $n \geq 1$, $S(n) \Rightarrow S(n+1)$.

  Then S(n) is true for all $n \geq 1$.
- **Proof by contradiction.**
  - Suppose all our assumptions are true, but S(n) is *not* true for all $n \geq 1$.
  - Let *k* be the smallest value of *n* such that S(k) is false.
  - $k \neq 1$ – why?
  - So $k > 1$. What's the contradiction?

# Template for using PMI in proofs.

- **Theorem**. S(n) is true for all $n \geq 1$.
- **Proof by Mathematical Induction.**
  - **Base case**: Prove S(1) is true.
  - **Inductive case**:
    - *Assume* $n \geq 1$ and S(n) is true.
    - *Prove* S(n+1) is true.
  - **By the Principle of Mathematical Induction, S(n) is true for all $n \geq 1$.** ∎
- The assumption "$n \geq 1$ and S(n) is true" is called the ***inductive assumption.***

# Example 1

- **Theorem**. For all $n \geq 1$, $5^n - 1$ is divisible by 4.
- **Proof by Mathematical Induction.**
  - **Base case.** Prove the theorem is true when $n = 1$.
  - **Inductive case.**
    - ***Assume:*** $n \geq 1$, and $5^n - 1$ is divisible by 4.
    - ***Prove:*** $5^{n+1} - 1$ is divisible by 4.
  - **Conclude**: The theorem is true by the Principle of Mathematical Induction. ∎

# Example 2

- **Theorem**. For all $n \geq 1$, $n < 2^n$.
- **Proof by Mathematical Induction.**
  - **Base case.** Prove the theorem is true when $n = 1$.
  - **Inductive case.**
    - ***Assume:*** $n \geq 1$, and $n < 2^n$.
    - ***Prove:*** $n+1 < 2^{n+1}$.
  - **Conclude**: The theorem is true by the Principle of Mathematical Induction. ∎

# More examples: Summations

- The symbol Σ means "summation" in math.

- $\sum_{i=1}^{n} f(i)$ means "the summation of f(i) from i = 1 to n."

- $$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \ldots + f(n-1) + f(n)$$

- $$\sum_{i=1}^{7} i = ??$$ $$\sum_{k=0}^{4} \left(\frac{1}{2}\right)^{k} = ???$$

# Summation Examples

- Prove:

- $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- $$\sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} = 2 - \left(\frac{1}{2}\right)^{n}$$