

MC215: MATHEMATICAL REASONING AND DISCRETE STRUCTURES

□ Friday, 10/3/08

■ From last time:

- Proving functions are one-to-one or onto

■ Today:

- Bijections and inverses
- Composition of functions

□ READING:

Still 3.1

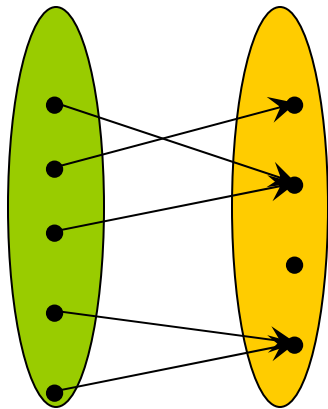
□ EXERCISES:

- pp. 132-134:
35, 36, 38, 39,
49, 51, 62, 63,
64, 89

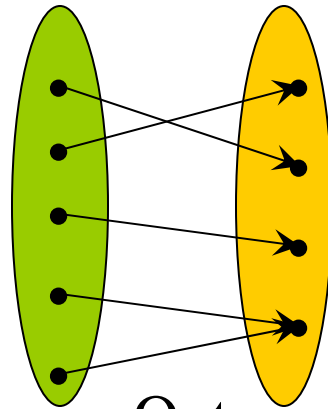
Bijections

- A function $f: A \rightarrow B$ is a **one-to-one correspondence**, or a **bijection**, iff it is both one-to-one and onto.
- A bijection is a pairing of the elements of A and B
 - **Onto:** Every b in B is $f(a)$ for at least one a in A
 - **One-to-one:** Every b in B is $f(a)$ for at most one a in A
 - **Together:** Every b in B is $f(a)$ for **exactly one** a in A

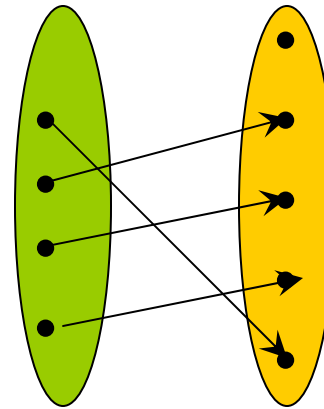
Illustrating Bijections, non-bijections



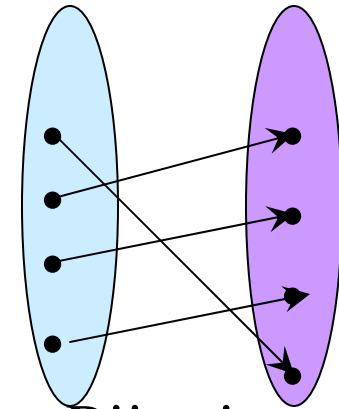
Not Onto
Not 1-1



Onto
Not 1-1



1-1
Not Onto



Bijection:
1-1
Onto

- Example: Prove that $f(x) = 3x - 5$ is a bijection from \mathbf{R} to \mathbf{R} .

Combining functions: Inverses

- If $f : A \rightarrow B$ is a bijection, then we can define a new function that 'reverses the arrows'
- This function is called the *inverse of f*
 - It can be defined **only** if f is a bijection
- Notation is $f^{-1} : B \rightarrow A$
 - This notation does **not** mean $1 / f$
- *Definition: $f^{-1}(b) =$ the unique a in A such that $f(a) = b$. Said another way:*

$$f^{-1}(b) = a \text{ IFF } f(a) = b$$

Finding inverses

- Remember: f must be a bijection!!
- Given a formula for $f(a)$
 - Set $b = f(a)$, and try to solve for a . The answer will be $f^{-1}(b)$... if it exists $\forall b$... and if it's unique!
 - Ex: Find the inverse of $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x - 5$
 - Ex: $f : \mathbb{N} \rightarrow \mathbb{N}, f(a) = a^2$
 - Ex: $f : \mathbb{N} \rightarrow \mathbb{N}, f(a) = 2a$
- If f is given as a list of values for a finite domain, just reverse each pair
 - Ex: $f : \{a, b, c\} \rightarrow \{x, y, z\},$
 $f(a) = y, f(b) = x, f(c) = z$

Combining functions: Composition

- Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$
 - Given $a \in A$, we can compute $b = f(a) \in B$
 - Since $b \in B$, we can now compute $g(b) = g(f(a)) \in C$
 - This is called the *composition of g with f* , denoted

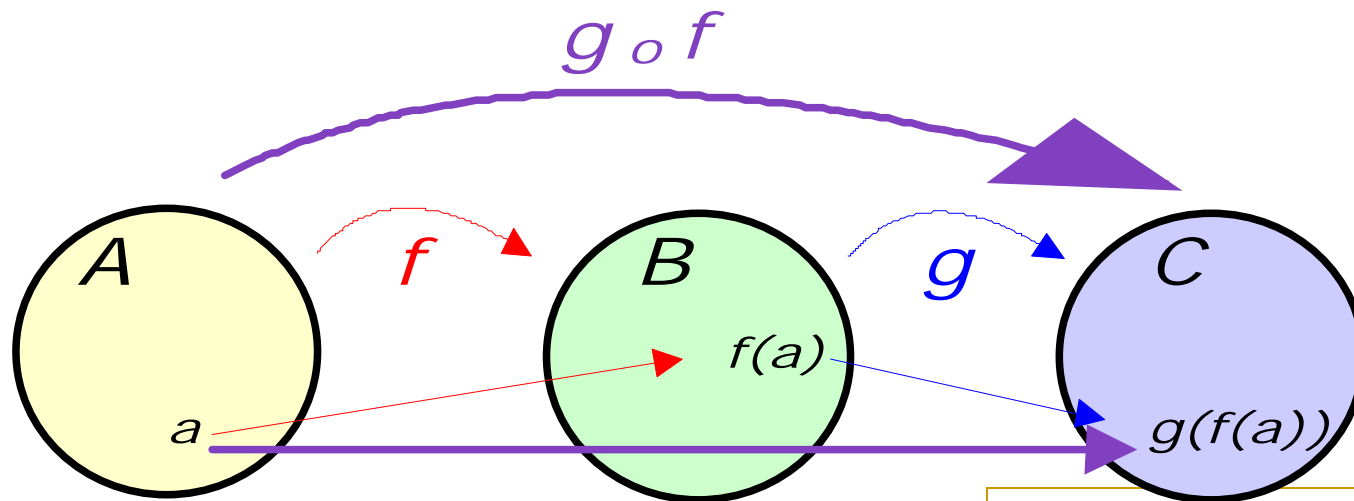
$$g \circ f : A \rightarrow C$$
$$g \circ f(a) = g(f(a))$$

Composition Arrow Diagram

$$g \circ f : A \rightarrow C$$

$$g \circ f(a) = g(f(a))$$

- Read the 'o' as 'of' or as 'after': 'g of f(a)' or 'g after f'



Composition Example

- Let $f : \mathbf{Z} \rightarrow \mathbf{N}, f(z) = 3z^2$
- Let $g : \mathbf{N} \rightarrow \mathbf{Z}, g(n) = n - 5$
- Let $h : \mathbf{Z} \rightarrow \mathbf{Z}, g(n) = 4 - n$
- Compute each of the following compositions (giving domain and codomain), or explain why you can't.
 - 1) $f \circ g$ 2) $g \circ f$ 3) $g \circ g$ 4) $h \circ g$
- What do your answers to 1 & 2 say about a *commutative law for composition*?
- 5) Compute h^{-1} , and then compute:
 - 6) $h \circ h^{-1}$ 7) $h^{-1} \circ h$

The Identity Function

- The function $I_A : A \rightarrow A, I(a) = a$, is called the *identity function of A*.
- Suppose $f : A \rightarrow B$
 - What is $f \circ I_A$
 - What is $I_B \circ f$
- Now suppose that f is a *bijection* from A to B
 - What is $f \circ f^{-1}$
 - What is $f^{-1} \circ f$