MC215: MATHEMATICAL REASONING AND DISCRETE STRUCTURES

- **Friday, 10/17/08**
  - From last time:
    - **Strings**
  - Today:
    - **Relations**
- ***HAND-IN HOMEWORK #3 WILL BE ON THE WEB LATER TODAY***
- **READING**: 3.3
- **EXERCISES**:
  - pp. 158: 20-23, 32 (see text for defs. of R^{-1} and partial order)

Relations on sets – the basic idea

Let S = {1, 2, …, 10}

- **Ex. 1 – a *function* as a relation:**
  - f: S ⟶ S, $f(s) = \lfloor s^{1/2} \rfloor$
  - f(1) = 1, f(5) = 2, etc.
  - We could say 2 is related to 5 (since f(2) = 5)
  - But 1 is *not* related to 7 (since f(1) ≠ 7)
- **Ex. 2 – *order* as a relation:**
  - The operator "≤ " on S
  - 3 ≤ 5, but 4 ≰ 1
  - We could say "3 is related to 5 (by ≤)"
- **Ex. 3: Sharing a particular property (*equivalence*)**
  - On S, the property of "having the same **parity**"
  - 1 has the same parity as 9, 6 has the same parity as 2
  - 6 is related to 2(by parity), but 3 is not related to 4

Relation – formal definitions

- **Def. 1:** A ***relation R on a set S*** is any set of ordered pairs from S.
  - I.e., a relation **R is any subset of S x S**.
- **Ex:** With S = {a, b, c, d, e}, define R = {(a, c), (b, d), (c, e), (a, e) }
  - We say "a is related to c", or "**a R c**," but c is not related to a – order matters.
- **Def. 2:** A ***relation R from a set S to a set T*** is any set of ordered pairs in S x T.
- **Ex:** R from **Z** to $\mathcal{P}$(**Z**) by: (n, A) $\in$ R if and only if n $\in$ A.
  - (3, {1, 3, 5}) $\in$ R, but (4, {1, 3, 5}) $\notin$ R

Some common examples

- For S = **R** (the real numbers), the relations
  - $=$, $\leq$, $\nleq$, $<$
- For S = $\mathcal{P}$(**R**) (all subsets of **R**)
  - $=$, $\subseteq$, "same size"
- For S = $\mathbf{N}^+$ (the positive integers)
  - a | b (a divides b), "same parity," "relatively prime"
- For S a set and $\mathcal{P}$(S) its power set
  - $\in$, $\notin$
- For S and T any sets, and f: S $\longrightarrow$ T a function
  - R = $\{(s, t) \mid s \in S, t \in T, t = f(s)\}$
  - Any function-relation has the property that for each $s \in S$, there is a unique ordered pair $(s, t) \in R$

Some properties of relations on S (not S x T, where S ≠ T)

- A relation R on S is ***reflexive*** if, for all s ∈ S, **s R s, i.e. (s, s) ∈ R.**
  - On previous slide, which relations are reflexive? (first 3 examples only)
- A relation R on S is ***symmetric*** if, for all s, t ∈ S, **s R t if and only if t R s.**
  - On previous slide, which relations are symmetric?
- A relation R on S is ***transitive*** if, for all s, t, u ∈ S, **if s R t and t R u, then s R u.**
  - On previous slide, which relations are transitive?

Antisymmetry

- Consider $\leq$ on **R**:
  - It's reflexive: $x \leq x$, for all x.
  - It's transitive: For all x, y, z, if $x \leq y$ and $y \leq z$, then $x \leq z$.
  - It's *not* symmetric: $x \leq y$ does not imply $y \leq x$.
  - In fact, if $x \leq y$ and $y \leq x$, then $x = y$.
- A relation on R is ***antisymmetric*** if, for all s, $t \in S$, **if s R t and t R s then s = t.**
  - On previous slide, which relations are antisymmetric?