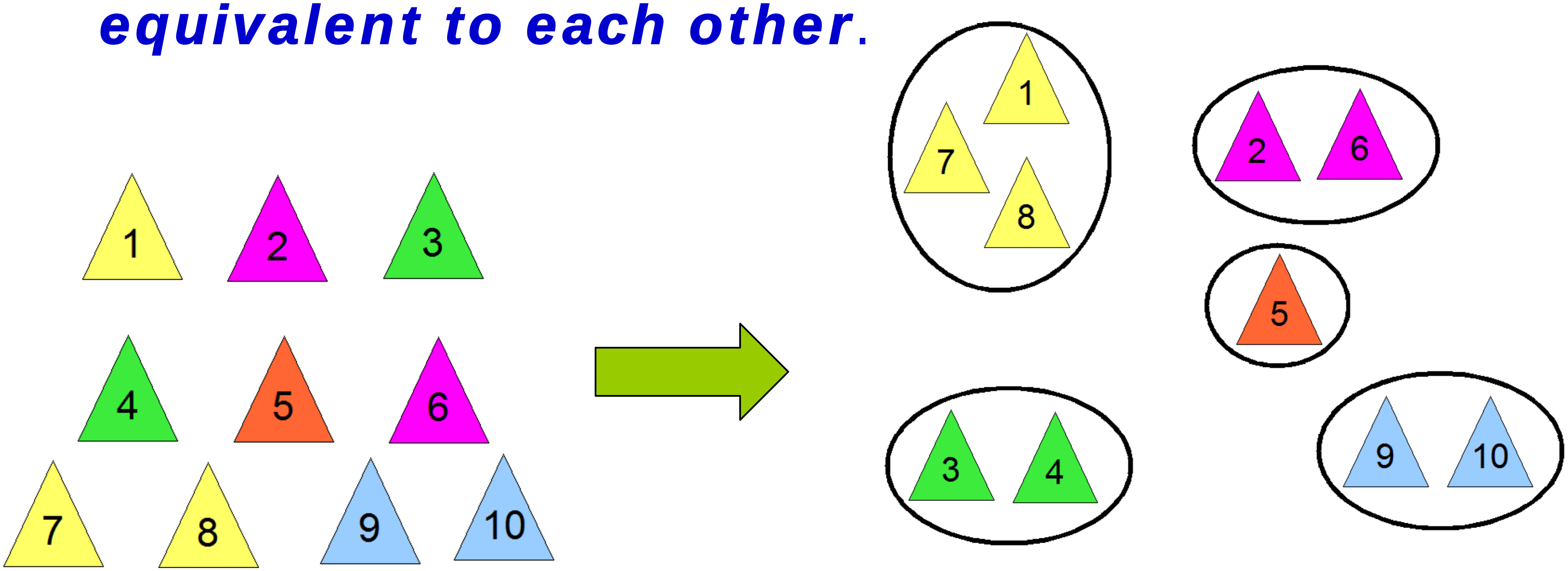
MC215: MATHEMATICAL REASONING AND DISCRETE STRUCTURES

- **Wednesday, 10/22/08**
  - From last time:
    - **Partial orders**
  - Today:
    - **Equivalence relations**
- **READING**: 3.4
- **EXERCISES**:
  - pp. 164-166: 1, 2, 9, 10, 15, 16, 38

Two ways to think of an equivalence relation on a set S

- A ***partition*** of S into ***pairwise disjoint*** subsets, S_1, ..., S_k, whose ***union is S***.
- Each subset is called an ***equivalence class***, and all elements in that subset are considered ***equivalent to each other***.



A second way to think of an equivalence relation on a set S

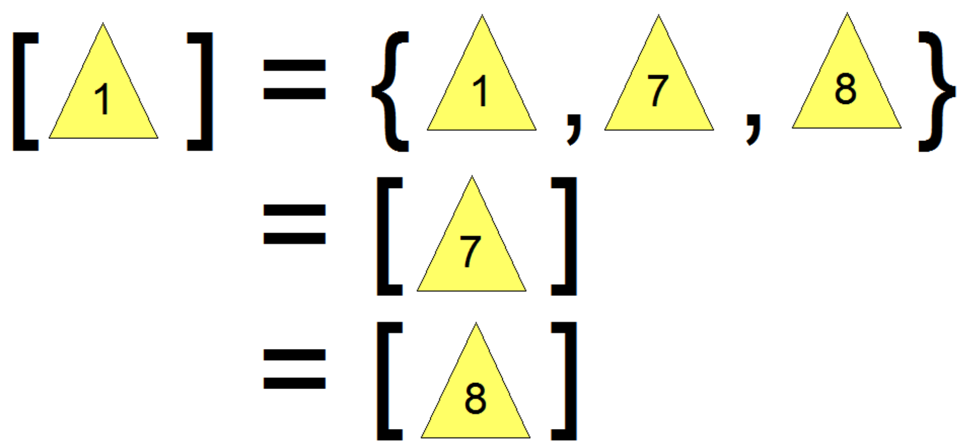
- An **equivalence relation** on a set S is a relation that is ***reflexive, symmetric, and transitive.***
- **Example:** For S on previous slide, define a relation by two triangles are equivalent if and only if they have the same color.
  - Reflexive?
  - Symmetric?
  - Transitive?

Example results from last time: Which are equivalence relations?

Relation	Reflexive	Symmetric	Anti-symmetric	Transi-tive	Partial Order	Equivalence Relation
On **R:** $x \le y$	YES	NO	YES	YES	**YES**	
On $\mathcal{P}$(**R**): $\subseteq$	YES	NO	YES	YES	**YES**	
On **N+:** a | b	YES	NO	YES	YES	**YES**	
On **R:** $x = y$	YES	YES	NO	YES	NO	
On $\mathcal{P}$(**R**): =	YES	YES	NO	YES	NO	
On **N+:** "same parity"	YES	YES	NO	YES	NO	
On **R:** $x > y$	NO	NO	NO	YES	NO	
On **N+:** "relatively prime"	NO	YES	NO	NO	NO	

From equivalence relation to partition

- If R is an equivalence relation, define a partition of S by putting two elements in the **same subset** if and only if they are **equivalent to each other**.
- Notation: **[a]** denotes the ***equivalence class of a***.
- In example:
  We call each element a ***representative*** of its class.



Theorem: Equivalence relation induces partition

- **Theorem.** If R is an equivalence relation on S, and we define, for each $a \in S$,
  $[a] = \{b \in S \mid (a, b) \in R\}$,
  then the collection of all the equivalence classes is a partition of S.
- **Proof.**
  - 1. Show any two equivalence classes are the same or are disjoint. We do this by showing:
    - **If $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$.**
  - 2. Show that every element of S is in some equivalence class.

From partition to equivalence relation

- Given a partition of S into subsets $S_1, S_2, \ldots, S_k$, define a relation R on S by **$(a,b) \in R$ if and only $a,b \in S_k$ for some k**
- **Theorem.** The relation defined above is an equivalence relation.
  - Reflexive? Symmetric? Transitive?
- The two processes,
  - Equivalence relation → partition, and
  - Partition → Equivalence relation

  are *inverse* operations.

Equivalence classes; congruence mod n

- Observation about equivalence classes:
  - If $(x, y) \in R$, then $[x] = [y]$. Why?
  - If $(x, y) \notin R$, then $[x] \cap [y] = \emptyset$. Why?
- **Congruence mod n on Z**.
  - Let n be a positive integer, and $a, b \in$ **Z.**
  - We say “a is congruent to b modulo n” if a – b is divisible by n.
  - We write $a \equiv b \pmod{n}$.
  - Said another way: **$a \equiv b \pmod{n}$ if and only if there is an integer k such that $a - b = k\,n$.**
- **Theorem.** Congruence mod n is an equivalence relation.

Example and properties of the "modulo n" equivalence relation

- **Example:** If n = 4, what are the equivalence classes induced by the "modulo 4" equivalence relation?
- **Theorem.** If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then
  - $a + c \equiv b + d \pmod{n}$
  - $a\, c \equiv b\, d \pmod{n}$
- We defined, as a *function* on Z, a mod n = remainder when a is divided by n.
- **Theorem**. $a \equiv b \pmod{n}$ if and only if a mod n = b mod n.