1. Use a proof by strong induction to prove that every integer \( n \geq 1 \) can be expressed as a sum of distinct powers of 2, i.e., any \( n \geq 1 \) can be expressed as follows:

\[
n = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r}, \text{ and no two of the powers, } p_1, \ldots, p_r, \text{ are the same.}
\]

For example, \( 25 = 2^4 + 2^3 + 2^0 \) (the powers 4, 3, and 0 are all different), and \( 154 = 2^7 + 2^4 + 2^3 + 2^1 \) (the powers 7, 4, 3, and 1 are all different). \textbf{Hint:} Notice that if you know how to express a number \( m \) this way, then you can easily figure out how to express \( 2m \) this way. For instance, if we want to write 50 this way, we can use what we know about 25:

\[
25 = 2^4 + 2^3 + 2^0, \text{ which implies that } 50 = 2(25) = 2(2^4 + 2^3 + 2^0) = 2^5 + 2^4 + 2^1.
\]

Use this idea for the inductive case, doing two cases depending on whether \( n + 1 \) is even or odd.

\textbf{SOLUTION TO #1: (10 pts)}

We give a proof by strong induction that the claim is true for all integers \( n \geq 1 \). For the base case, note that \( 1 = 2^0 \). For the inductive case, assume that \( n \geq 1 \), and that every integer from 1 through \( n \) can be expressed as a sum of distinct powers of 2. We must show that \( n + 1 \) is a sum of distinct powers of 2. We do this in two cases, depending on whether \( n + 1 \) is even or odd.

\textbf{Case 1, } \( n + 1 \text{ is even:} \) In this case \((n + 1)/2\) is an integer. Since \( n + 1 \) is a positive integer, it follows that \((n + 1)/2\) is an integer between 1 and \( n \), so by the inductive hypothesis, \((n + 1)/2 = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r}\), and no two of the powers, \( p_1, \ldots, p_r \), are the same. Multiplying both sides of this equation by 2 yields

\[
n + 1 = 2(2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r}) = 2^{p_1+1} + 2^{p_2+1} + 2^{p_3+1} + \cdots + 2^{p_r+1},
\]

and no two of the powers, \( p_1 + 1, \ldots, p_r + 1 \), are the same, since adding 1 to a set of pairwise distinct numbers maintains that property. Thus the claim holds when \( n + 1 \) is even.

\textbf{Case 2, } \( n + 1 \text{ is odd:} \) In this case \( n \) is even, and by the inductive hypothesis, \( n = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r} \), and no two of the powers, \( p_1, \ldots, p_r \), are the same. Since \( n \) is even it is also true that none of the powers, \( p_1, \ldots, p_r \), is 0, because otherwise \( n \) would be a sum of even numbers plus \( 1 = 2^0 \), which would make \( n \) odd. Hence, \( n + 1 = n + 2^0 = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r} + 2^0 \), and no two of the powers, \( p_1, \ldots, p_r, 0 \), are the same. Thus the claim holds when \( n + 1 \) is odd.

It follows by the Principle of Mathematical induction that every positive integer \( n \) can be expressed as a sum of distinct powers of 2.

\textbf{Alternate way to do the inductive case (adapted from Ria Bhatnagar’s solution):} For the inductive case, assume that \( n \geq 1 \), and that every integer from 1 through \( n \) can be expressed as a sum of distinct powers of 2. We must show that \( n + 1 \) is a sum of distinct powers of 2. Let \( 2^k \) be the largest power of 2 such that \( 2^k \leq n + 1 \). If equality holds we’re done. If \( 2^k < n + 1 \), then let \( m = n + 1 - 2^k \). Then \( 1 \leq m \leq n \), so by the inductive hypothesis, \( m = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r} \), where all the powers are different. Thus \( n + 1 = m + 2^k = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots + 2^{p_r} + 2^k \). We
must show the powers are still all different, i.e., there is no $i$ such that $2^{p_i} = 2^k$. If that were true then we would have $n + 1 = 2^{p_1} + 2^{p_2} + 2^{p_3} + \cdots 2^k + \cdots + 2^{p_r} + 2^k \geq 2^k + 2^k = 2^{k+1}$, which contradicts the assumption that $2^k$ is the largest power of 2 such that $2^k \leq n + 1$.

2. Recall that a tree is a graph that is connected and has no cycles. In class we gave a proof using strong induction that any tree $T$ with $n \geq 1$ vertices has exactly $n - 1$ edges. In this problem you will first prove a lemma (a result that helps prove the main theorem), and then you will give a proof using weak induction that any tree $T$ with $n \geq 1$ vertices has exactly $n - 1$ edges.

a) Prove the following lemma:

**Lemma.** If $T$ is a tree with two or more vertices, then $T$ has at least two vertices of degree 1 (the degree of a vertex is the number of edges it is on).

*Hint:* A path in a graph is a sequence of vertices, each one adjacent to the next, and with no repeated vertices. For example, in the tree shown here, 6-8-2-9-7 is a path, and so is 3-2-9-4, but 3-2-7-9 is not (2 and 7 are not adjacent), nor is 1-9-4-9-2-3 (9 is repeated). A longest possible path is one that cannot be extended any further. For example, 6-8-2-9-7 is not longest possible, because it could be extended from 7 to 5. But 3-2-9-4 is a longest possible path. To prove the lemma, let $T$ be a tree with two or more vertices, and let $P$ be a longest possible path in $T$. Prove that the two endpoints of $P$ both must have degree 1.

b) Use the lemma from part (a) to give a proof by (weak) induction on the number $n \geq 1$ of vertices in a tree $T$ that $T$ has exactly $n - 1$ edges.

*Hint:* For the weak inductive assumption you'll assume that $n \geq 1$, and that the claim is true for any tree with $n$ vertices. To prove the inductive conclusion, let $T$ have $n + 1$ vertices. To use the inductive assumption, you should say how to remove a vertex of $T$ to get a tree with $n$ vertices. But you need to be careful what vertex you remove, so that what's left is still a tree. For example, in the tree shown above, you can delete vertex 4 or 5 and still have a tree, but if you delete vertex 8 or 9, what's left is not a tree, because it's not connected. Think about how the lemma from part (a) can help you here.

**SOLUTION TO #2:** (14 pts: a-6, b-8)

a. Suppose $T$ is a tree with two or more vertices. Let $P$ be any "longest path" in $T$, i.e. a path that is not part of any longer path. Since $T$ has two or more vertices and is connected, every vertex has degree $\geq 1$. Thus $P$ has two or more vertices, i.e., $P = v_1 \sim v_2 \sim \cdots \sim v_n$, where $n > 1$ (so $v_1 \neq v_n$), and the notation $v \sim w$ indicates that the two vertices $v$ and $w$ are adjacent. We claim that $v_1$ has degree 1, i.e., its only neighbor is $v_2$. Suppose not, i.e., $v_1$ has another neighbor $w \neq v_2$. The vertex $w$ cannot be on the path $P$, because then $v_1 \sim v_2 \sim \cdots w = v_l \sim v_1$ forms a cycle, contradicting the assumption that $T$ is a tree. But if $w$ is not on the path $P$, then we can extend $P$ to form the longer path $w \sim v_1 \sim v_2 \sim \cdots \sim v_n$, contradicting the assumption that $P$ is a longest path. This proves that
\(v_1\) has degree 1. The same argument proves that \(v_n\) has degree 1, so that \(T\) has at least two vertices of degree 1, as claimed.

\[\blacksquare\]

b. We use (weak) induction on \(n \geq 1\) to prove that any tree with \(n\) vertices has exactly \(n - 1\) edges. The base case, \(n = 1\), is obvious, since in this case \(T\) has 0 edges. For the inductive case, assume that \(n \geq 1\), and that any tree with \(n\) vertices has \(n - 1\) edges. Let \(T\) be a tree with \(n + 1\) edges. Since \(n + 1 \geq 2\), it follows from the lemma that \(T\) has at least two degree-1 vertices. Let \(v\) be a degree-1 vertex of \(T\), with neighbor \(w\). If we delete the vertex \(v\) and the edge \(v \sim w\) (but not the vertex \(w\)), we do not disconnect the tree and we do not introduce any cycles. Thus the resulting graph is a tree \(T'\) with \(n\) vertices and one fewer edge than \(T\). By the inductive hypothesis, \(T'\) has exactly \(n - 1\) edges, and so \(T\) has \(n - 1 + 1 = n\) edges, proving the inductive case. By the Principle of Mathematical Induction, the claim holds for all trees with \(n \geq 1\) vertices.

\[\blacksquare\]

3.

a) For each of the three functions given below, do the following: (i) say what the domain, codomain, and range of the function are; (ii) say whether the function is onto, and justify your answer; (iii) say whether the function is one-to-one, and justify your answer.

i) \(f: \mathbb{R} \to \mathbb{R}, f(x) = 12x - 23\)

ii) \(g: \mathbb{R} \to \mathbb{Z}, g(x) = \lceil x + 4 \rceil\) (the ceiling function rounds up to the nearest integer)

iii) \(h: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}, h(n) = (2n + 3) \mod 9\)

b) For each of the functions in part (a), if the function is not both one-to-one and onto, modify its domain and codomain, but not its formula, so that the modified function is both one-to-one and onto (if the given function is one-to-one and onto, just say that).

Here is an example to clarify what is wanted in this question: The function \(k: \mathbb{Z} \to \mathbb{Z}, f(x) = x^2\) is not one-to-one (e.g., \(k(-2) = k(2) = 4\)), nor is it onto (e.g., \(-3\) is not in the range of \(k\)). But the function \(k^*: \mathbb{N} \cup \{0\} \to \{0, 1, 4, 9, \ldots, n^2, \ldots\}, f(x) = x^2\), which has the same formula as \(k\) but different domain and codomain, is both one-to-one and onto.

Note: The original and modified functions should have the same range.

SOLUTION TO #3: (39 pts: a-30 (10 per function); b-9 (3 per function))

a. \(f: \mathbb{R} \to \mathbb{R}, f(x) = 12x - 23\)

- **Domain** and **codomain** are both \(\mathbb{R}\).
- **Range** is \(\mathbb{R}\) because \(f\) is onto, which is proved below.
**f is onto.** Proof: Let \( y \) be any element of \( \mathbb{R} \). We must find some \( x \in \mathbb{R} \) such that \( y = f(x) \). Using the definition of \( f \), we get \( y = f(x) \Leftrightarrow y = 12x - 23 \Leftrightarrow \frac{y-23}{12} = x \). Hence for any \( y \in \mathbb{R} \), we have shown that \( f \left( \frac{y-23}{12} \right) = y \). This proves that \( f \) is onto.

**f is one-to-one.** Proof: Suppose \( f(a) = f(b) \). We must prove that \( a = b \). Using the definition of \( f \), we get: \( f(a) = f(b) \Leftrightarrow 12a - 23 = 12b - 23 \). Adding 23 to both sides and then dividing both sides by 12 yields \( a = b \), proving that \( f \) is one-to-one.

\[
g: \mathbb{R} \to \mathbb{Z}, \, g(x) = \lfloor x + 4 \rfloor
\]

**Domain** is \( \mathbb{R} \), **codomain** is \( \mathbb{Z} \).

**g is onto.** Proof: Let \( y \) be any element of \( \mathbb{Z} \). Then \( g(y - 4) = \lfloor y - 4 + 4 \rfloor = \lfloor y \rfloor = y \). This proves that \( g \) is onto.

**g is not one-to-one.** Proof: \( g(1.1) = g(1.2) = 2 \), which shows that \( g \) is not one-to-one.

\[
h: \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}, \, h(n) = (2n + 3) \mod 9
\]

**Domain** and **codomain** are both \( \mathbb{N} \cup \{0\} \).

**Range** is \( \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \). Since any integer "mod 9" is a remainder after dividing by 9, it must be an integer from 0 to 8, so no other value of \( h(n) \) is possible. We give specific examples to prove that every value from 0 to 8 is a value of this function: \( 0 = h(3), 1 = h(8), 2 = h(4), 3 = h(0), 4 = h(5), 5 = h(1), 6 = h(6), 7 = h(2), 8 = h(7) \). This proves the range is \( \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \).

**h is not onto.** Proof that: The integer \( y = 10 \) is an element of the codomain that is clearly not in the range of \( h \). This proves that \( h \) is not onto.

**h is not one-to-one.** Proof: \( h(0) = g(3) = 0 \), which shows that \( h \) is not one-to-one.

b. Modifying the domains and/or codomains to make the functions one-to-one and onto:

**f: \( \mathbb{R} \to \mathbb{R}, f(x) = 12x - 23 \):** This function is already one-to-one and onto, so no modification is needed.

**g: \( \mathbb{R} \to \mathbb{Z}, g(x) = \lfloor x + 4 \rfloor \):** The range of this function is \( \mathbb{Z} \), i.e., \( g \) is onto, so we don't change that. To make it one-to-one, we change the domain to also be \( \mathbb{Z} \). Then \( g(x) = \lfloor x + 4 \rfloor = x + 4 \), since \( x \) is an integer. On the domain \( \mathbb{Z} \), \( g(x) \) is one-to-one, because \( g(a) = g(b) \Leftrightarrow \lfloor a + 4 \rfloor = \lfloor b + 4 \rfloor \Leftrightarrow a + 4 = b + 4 \Leftrightarrow a = b \).

**h: \( \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}, h(n) = (2n + 3) \mod 9 \):** As shown above, the range of \( h \) is \( \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \). We must restrict the domain so that each value in the range is taken exactly once. The proof above also demonstrates that \( h \) takes on all these values exactly once if we let the domain be the same set as the range, \( \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \). There are other ways to correctly restrict the domain to make \( h \) one-to-one. For example, the set \( \{31, 32, 33, 34, 35, 36, 37, 38, 39\} \) also would be a valid domain.