Note

On the Thickness and Arboricity of a Graph

ALICE M. DEAN

Department of Mathematics and Computer Science,
Skidmore College, Saratoga Springs, New York 12866

JOAN P. HUTCHINSON*

Department of Mathematics, Macalester College,
St. Paul, Minnesota 55105

AND

EDWARD R. SCHEINERMAN†

Department of Mathematical Sciences, The Johns Hopkins University,
Baltimore, Maryland 21218

Communicated by the Editors

Received November 23, 1988

We prove that the thickness and the arboricity of a graph with \( e \) edges are at most \( \lceil \sqrt{\pi / 3} + 3/2 \rceil \) and \( \lceil \sqrt{\pi / 2} \rceil \), respectively, and that the latter bound is best possible. © 1991 Academic Press, Inc.

The thickness of a graph \( G \), \( \theta(G) \), is the minimum number of planar graphs into which the edges of \( G \) can be partitioned, and the arboricity, \( Y(G) \), is the minimum number of acyclic graphs into which the edges of \( G \) can be partitioned. Nash-Williams [9] has determined a precise formula for the arboricity of a graph; namely,

\[
Y(G) = \max \left\lfloor \frac{e_H}{n_H - 1} \right\rfloor,
\]

* Research supported in part by N.S.F. Grant RII-8901458 and by Smith College.
† Research supported in part by the Office of Naval Research contract N00014-85-K0622.

0095-8956/91 $3.00
Copyright © 1991 by Academic Press, Inc.
All rights of reproduction in any form reserved.
where the maximum is taken over all nontrivial induced subgraphs $H$, and where $e_H$ and $n_H$ denote the number of edges and vertices of $H$, respectively. From this and Euler’s formula for planar graphs it follows that $Y(G) \leq 3\theta(G)$; clearly $\theta(G) \leq Y(G)$.

First we derive the $O(\sqrt{e})$ bound on the thickness of a graph with $e$ edges, and then, using the Nash-Williams formula, we obtain the best possible bound of $\lceil \sqrt{e/2} \rceil$ on the arboricity, exhibiting, for each $e$, a graph with $e$ edges whose arboricity achieves this bound. It was previously known that $Y(G) = O(\sqrt{e})$ since in [5] it is shown that $Y(G) \leq \lceil (1/2) \sqrt{2e + n} \rceil$, where $n$ is the number of vertices in the graph, and in [7] that $Y(G) \leq \lceil 5/4 + (1/2) \sqrt{2e - 7/4} \rceil$. Both these bounds are achieved by the complete graphs; the latter bound is asymptotic to our bound, but infinitely often is larger by 1. We use definitions and basic facts from [2].

**Theorem 1.** If $G$ is a simple graph with $e$ edges, then $\theta(G) \leq \lceil \sqrt{e/3} + 3/2 \rceil$.

**Proof.** We use induction on $|V| + |E|$. If $|V| + |E| = 1$, then $G = K_1$ and $\theta(K_1) = 0$. Suppose the theorem has been proved for all graphs with $|V| + |E| < n + e$ and let $G$ be a graph with $n$ vertices and $e$ edges.

First, suppose there is a vertex $v$ with $\deg(v) \leq \lceil \sqrt{e/3} \rceil$. By induction, $G - v$ has thickness at most $\lceil \sqrt{e/3} + 3/2 \rceil$. Let $k = \lceil \sqrt{e/3} + 3/2 \rceil$ and decompose $G - v$ into $k$ planar graphs $H_1, \ldots, H_k$. Since $\deg(v) \leq k$, we add to each of the $H_i$, for $1 \leq i \leq \deg(v)$, the vertex $v$ and the edge from $v$ to its $i$th neighbor. Note that so modified, the $H_i$'s are planar graphs whose union is $G$.

On the other hand, suppose there is no vertex with degree at most $\lceil \sqrt{e/3} \rceil$. In this case,

$$2e = \sum_{v \in V(G)} \deg(v) > n \sqrt{e/3}$$

and therefore $n < 2\sqrt{3e}$. Since the thickness of $K_n$ is at most $\lceil (n + 9)/6 \rceil$ (see [1, 3, 8, 11, 12]), we have

$$\theta(G) \leq \theta(K_n) \leq \lceil \frac{n + 9}{6} \rceil \leq \lceil \sqrt{\frac{e}{3}} + \frac{3}{2} \rceil.$$

Since the thickness of the complete graph on $n$ vertices is $O(n)$, the bound of this result is of the right order, but we believe that the constants are not best possible. Note that $\theta(K_n)$ is approximately $\sqrt{e}/18$, but $\theta(K_{n/2,n/2}) = \lceil n^2/(18n - 16) \rceil$ is approximately $\sqrt{e}/16$ (see [4]). We conjecture that $\theta(G) \leq \sqrt{e}/16 + O(1)$ for any graph $G$. 

The previous proof with "arboricity" replacing "thickness" and the fact that \( Y(K_n) \leq \lceil n/2 \rceil \) leads to \( Y(G) \leq \lceil \sqrt{e} \rceil \). However, we prove the following stronger and best possible result.

**Theorem 2.** If \( G \) is a simple graph with \( e \) edges, then \( Y(G) \leq \lceil \sqrt{e/2} \rceil \), and this bound is best possible.

**Proof.** By Nash-Williams' result [9], there is a subgraph \( G' \) of \( G \) with \( n' \) vertices and \( e' \) edges, and

\[
Y(G) = Y(G') = \left\lfloor \frac{e'}{n'-1} \right\rfloor.
\]

If \( e' \leq (n'-1)^2/2 \) then \( (e'/(n'-1))-1 \leq e'/2 \), and

\[
Y(G) \leq \left\lfloor \frac{e'}{\sqrt{2}} \right\rfloor \leq \left\lfloor \frac{e}{\sqrt{2}} \right\rfloor.
\]

If \( e' > (n'-1)^2/2 \), we have

\[
Y(G) = Y(G') \leq Y(K_{n'}) = \left\lfloor \frac{n'}{2} \right\rfloor,
\]

and we compute

\[
e' > (n'-1)^2/2 = \sqrt{\frac{e'}{2}} > \frac{n'-1}{2}
\]

\[
= \left\lfloor \frac{\sqrt{e}}{2} \right\rfloor > \frac{n'-1}{2} \geq Y(G).
\]

Next, we show that the inequality is best possible in the sense that among all graphs with \( e \) edges there is one for which \( Y = \lceil \sqrt{e/2} \rceil \). The construction is as follows: Let \( n \) be the integer satisfying

\[
\left( \begin{array}{c} n \\ 2 \end{array} \right) \leq e < \left( \begin{array}{c} n+1 \\ 2 \end{array} \right).
\]

So put \( e = \left( \begin{array}{c} n \\ 2 \end{array} \right) + k \) with \( 0 \leq k < n \). We make a graph \( G \) with \( n+1 \) vertices. Form a complete graph on \( n \) of them and let the degree of the remaining vertex be \( k \). We claim that \( Y = Y(G) = \lceil \sqrt{e/2} \rceil \). We have \( Y \leq \lceil \sqrt{e/2} \rceil \) so we work to show \( Y \geq \lceil \sqrt{e/2} \rceil \). By the Nash-Williams result,

\[
Y \geq \max \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n+k}{n} \right\rfloor \right\}.
\]
Notice that
\[
\frac{\binom{n}{2} + k}{n} = \frac{n-1}{2} + \frac{k}{n}.
\]

Now if \(k/n \leq 1/2\) we have that \(Y \geq \lceil n/2 \rceil\) and
\[
\sqrt{\frac{2}{3}} = \sqrt{\frac{(n^2 + k)}{2}} \leq \sqrt{\frac{(n^2 + n/2)}{2}} = \sqrt{\frac{n^2}{4}} = \frac{n}{2}
\]
and therefore
\[
\lceil \sqrt{2/3} \rceil \leq \lceil \frac{n}{2} \rceil \leq Y.
\]

On the other hand, if \(k/n > 1/2\) then \(Y \geq \lceil (n + 1)/2 \rceil\) and
\[
\sqrt{\frac{2}{3}} \leq \sqrt{\frac{(n^2 + n - 1)}{2}} = \sqrt{\frac{n^2 + n - 2}{4}} \leq \frac{n + 1}{2}
\]
and therefore
\[
\lceil \sqrt{2/3} \rceil \leq \lceil \frac{n + 1}{2} \rceil \leq Y.
\]

If \(G\) is a triangle-free graph, we can modify the above proof and apply Turán's theorem to bound its arboricity by \((1/2) \sqrt{\bar{e}}\). This bound is achieved by the complete bipartite graph \(K_{n,n}\). This also gives a slightly improved thickness bound for triangle-free graphs.

The proof technique of Theorem 1 is a simplification of that used in [6] to show that the thickness and arboricity of a graph of genus \(g\) are \(O(\sqrt{g})\). This technique leads to a simpler proof of the result in [6] that \(\theta(G) \leq 6 + \sqrt{2g - 2}\). The proof technique of Theorem 2 is derived from that of [10] where the maximum interval number of a graph is compared with the genus of a graph.

**ACKNOWLEDGMENTS**

The authors thank a referee, whose comments simplified the proof of Theorem 2, and H. S. Wilf for several helpful comments on the manuscript.

**References**