Relations among Embedding Parameters for Graphs

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ABSTRACT

Let G be a simple graph with n vertices, and let a_1, bt, γ, and θ denote, respectively, the arboricity, book thickness (also called pagenumber), genus, and thickness of G. We establish inequalities, each of which is best possible up to a constant, between pairs of these parameters. In particular, we show:

$$\theta \le 6 + \sqrt{2\gamma - 2},$$
$$a_1 \le bt + 1, \quad \text{and}$$
$$\gamma \le (\theta - 1)(n - 1).$$

1. Introduction

There are several different ways to characterize the embeddability of a graph G. Some of the numerical measures include the genus γ, the thickness θ, the book thickness (or pagenumber) bt, and the arboricity a_1. These parameters have many applications, including VLSI design and complexity theory.

In this paper, we study relationships between these parameters. We prove several inequalities, each of which is best possible up to a constant.

2. Definitions and Basic Properties

Throughout, G will denote a simple, *connected* graph, and n and e (or n_G and e_G, if more than one graph is being considered) will denote, respectively, the number of

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vertices and edges of G. We will consider four parameters that measure the embeddability of G in different ways.

First, the (orientable) *genus of G*, denoted by γ, is the minimum number k such that G can be embedded on a sphere with k handles. Second, the *thickness of G*, denoted by θ, is the minimum number of edge–disjoint planar subgraphs whose union is G. Third, the *arboricity of G*, denoted by a_1, is the minimum number of edge–disjoint forests whose union is G. Finally, the *book thickness* or *pagenumber of G*, denoted by bt, is the minimum number of pages joined at a common spine such that G can be embedded with its vertices on the spine and edges on the pages.

All these parameters are known for the complete graphs, the complete bipartite graphs, and the generalized cubes. For example, $\gamma(K_n) = \lceil (n-3)(n-4)/12 \rceil$ for $n \geq 3$, $\theta(K_n) = \lfloor (n+7)/6 \rfloor$ (except that $\theta(K_9) = \theta(K_{10}) = 3$), $a_1(K_n) = \lceil n/2 \rceil$, and $bt(K_n) = \lceil n/2 \rceil$ for $n \geq 4$ (see [BCL, BK]).

There are two fundamental results concerning these parameters that will be used throughout the paper. The first is Euler's formula (and a corollary), and the second is Nash–Williams' formula for arboricity:

Theorem 1 *If G is embedded on S_g, the sphere with g handles, then, letting f denote the number of faces:*

(i) $n - e + f \geq 2 - 2g$, *with equality if and only if the embedding is a 2–cell embedding,*

(ii) $e \leq 3n - 6 + 6g$.

Theorem 2 [N] *For any graph G,*

$$a_1 = \max_{H < G} \left\lceil \frac{e_H}{n_H - 1} \right\rceil,$$

where H ranges over all nontrivial induced subgraphs of G.

It follows easily from Theorem 2 that a_1 is bounded in terms of the minimum degree δ and maximum degree Δ of G:

Corollary 2 *(i)* $\left\lceil \frac{\delta+1}{2} \right\rceil \leq a_1 \leq \left\lceil \frac{\Delta+1}{2} \right\rceil$.

(ii) [AH] *If G is d–regular, then* $a_1 = \left\lceil \frac{d+1}{2} \right\rceil = \left\lceil \frac{e}{n-1} \right\rceil$.

The arboricity and thickness of a graph G are closely related in size; namely, $\theta \leq a_1 \leq 3\theta$. The first inequality follows from the definitions, and the second follows from Theorems 1 and 2, which together imply that any planar graph has arboricity at most 3. Furthermore, these bounds are sharp, in the sense that for any θ, there are graphs that achieve each extreme; see [T, BH].

Another related parameter is the *vertex arboricity*, denoted by a. The vertex arboricity of a graph G is the minimum number of vertex–disjoint forests whose union is G. In [B] it is shown that $a \le a_1$, and so any bound on arboricity will give the same bound for vertex arboricity.

3. Upper Bounds on Thickness and Arboricity

In [A], Asano showed that triangle–free graphs and toroidal graphs have thickness $\theta \le \gamma + 1$. A recent result of Scheinerman [S] shows that for any G, the arboricity a_1 satisfies $a_1 \le 2 + \sqrt{3\gamma}$. Since $\theta \le a_1$, it follows that, for any G, $\theta \le 2 + \sqrt{3\gamma}$. Archdeacon and Richter (unpublished manuscript) have obtained a similar bound on θ. Furthermore, the example of the complete graph K_n shows that this result is best possible up to a constant, since $\theta(K_n) \approx \sqrt{\gamma(K_n)/3}$.

Using an idea due to M. Albertson and C. Thomassen (personal communication), we improve on Scheinerman's result for thickness, reducing the constant from $\sqrt{3}$ to $\sqrt{2}$.

Theorem 3 *If G is a simple graph, then*
$$\theta \le 6 + \sqrt{2\gamma - 2}.$$

Proof Let $k = 6 + \lfloor \sqrt{2\gamma - 2} \rfloor$ and let $G_1, G_2, \ldots, G_k$ each be graphs with the same vertex set as G, but initially no edges. For each vertex v with $\deg(v) \le k$, remove its edges from G and place one edge in each G_i for $1 \le i \le \deg(v)$. Repeat this process for each vertex of degree at most k in the resulting graph until all the remaining vertices in the final graph G' have degree strictly greater than k.

Each of the graphs G_i generated by this process is acyclic: Suppose some G_i contained a cycle C. Consider the "oldest" vertex v on the cycle C, i.e., the vertex on C whose edges were removed from G before any other vertices of C. Then by the construction of the G_i's, the degree of v in C is 0 or 1, a contradiction.

Since all remaining vertices of G' have degree at least $k + 1 \ge 6 + \sqrt{2\gamma - 2}$, Thm. 1 (ii) yields
$$(6 + \sqrt{2\gamma(G) - 2})\, n' \le 2e' \le 6n' + 12(\gamma(G') - 1),$$
and, since $\gamma(G') \le \gamma(G)$,
$$(6 + \sqrt{2\gamma(G) - 2})\, n' \le 2e' \le 6n' + 12(\gamma(G) - 1),$$
where n' and e' are the number of vertices and edges of G'. Thus,
$$n' \le 6\sqrt{2\gamma(G) - 2}.$$

Now G' is a subgraph of the complete graph $K_{n'}$, which satisfies
$$\theta(K_{n'}) = \begin{cases} \lfloor (n' + 7) \rfloor / 6, & n' \ne 9, 10 \\ 3, & n' = 9, 10 \end{cases}$$

It follows that

$$\begin{aligned}\theta(G') &\le \theta(K_{n'}) \\ &\le \lfloor (n' + 9)/6 \rfloor \\ &\le \lfloor (6\sqrt{2\gamma(G) - 2} + 9)/6 \rfloor \\ &\le \sqrt{2\gamma(G) - 2} + 6.\end{aligned}$$

Hence, G' can be partitioned into at most $k = \lfloor \sqrt{2\gamma(G) - 2} \rfloor + 6$ planar subgraphs, say $P_1, \ldots, P_{k'}$, where $1 \le k' \le k$. Form the unions $P_i \cup G_i$ for $i = 1, \ldots, k'$.

We claim that each tree of G_i is incident with at most one vertex of P_i and hence $P_i \cup G_i$ is a planar graph. If not, find a tree T_i of G_i and two vertices x and y of $P_i \cap T_i$ such that the path P_{xy} from x to y in T_i contains no other vertex of P_i. Note that x and y are not adjacent since each edge of G_i is incident with at least one vertex not in P_i. As before, let z be the "oldest" internal vertex of P_{xy}. Its degree in P_{xy} is 0 or 1, a contradiction.

In summary, the original graph G has thickness at most k, where $k \le 6 + \sqrt{2\gamma(G) - 2}$. ❑

Since $a_1 \le 3\theta$, we obtain as an immediate corollary that $a_1(G) \le 3\sqrt{2\gamma(G) - 2} + 18$. However, we can improve the constant if, in the proof of Theorem 3, we replace "thickness" by "arboricity" and use the arboricity of $K_{n'}$ to estimate $a_1(G')$. The resulting inequality is weaker than Scheinerman's best possible result that $a_1(G) \le 2 + \sqrt{3\gamma(G)}$ (cf. [S]):

Corollary 3 *If G is a simple graph, then*

$$a_1(G) \le 6 + \sqrt{6\gamma(G) - 6}.$$

The examples of the complete graph, the complete bipartite graph, and the generalized cube all have $\theta \approx 2\gamma/n$, and this suggests a natural bound on θ in terms of γ and n. Since for all graphs $\gamma \le n^2/12$, we have $\gamma/n \le \sqrt{\gamma}$, suggesting a tighter upper bound: $\theta = O(\gamma/n)$. However, this bound does not hold in general: Let G be any graph with $\theta = O(\gamma/n)$, and embed G on its genus surface. Append an arbitrary number of vertices of degree one to G and note that the resulting graphs may no longer satisfy a $O(\gamma/n)$ bound, since n can become arbitrarily large. In contrast, in the next section we will show that $\gamma = O(\theta n)$ for all graphs.

However, for many graphs including the regular graphs it is true that $a_1 = O(\gamma/n)$ and hence $\theta = O(\gamma/n)$.

Theorem 4 *Let $0 < \varepsilon \le 1$. If G is a graph with at least $1 + 1/\varepsilon$ vertices and such that $a_1(G) = \lceil e/(n-1) \rceil$, then*

$$a_1(G) \le \lceil (3 + 6\gamma/n)(1 + \varepsilon) \rceil.$$

Proof By Theorem 1 (ii),

$$\begin{aligned} a_1(G) &= \lceil e / (n-1) \rceil \\ &\le \lceil (3n - 6 + 6\gamma) / (n-1) \rceil \\ &\le \lceil (3 + 6\gamma/n)(1 + 1/(n-1)) \rceil \\ &\le \lceil (3 + 6\gamma/n)(1 + \varepsilon) \rceil. \; \square \end{aligned}$$

From Corollary 2 (ii) we have:

Corollary 4 *Let $0 < \varepsilon \le 1$. If G is a regular graph with at least $1 + 1/\varepsilon$ vertices, then*

$$a_1(G) \le \lceil (3 + 6\gamma/n)(1 + \varepsilon) \rceil = O(\gamma/n).$$

Since $\theta \le a_1$, we get the same bound on θ.

In [H], Hutchinson shows that a graph G with a 2–cell embedding on the k–handled sphere S_k is 5-colorable if the embedding has "short" edges. There, the length of an edge is measured relative to the length of a side of the standard 4k–gon on which the embedding is represented. The following theorem shows that under those conditions, G has thickness at most 2 (and hence arboricity at most 6).

Theorem 5 *Suppose G has a 2–cell embedding on a surface of genus $k \ge 1$, and suppose that G has a representation G_k on the standard 4k–gon (i.e., a 4k–gon in which each side has unit length) such that every edge of G_k has length $< \varepsilon$, where ε is at most 1/2. Then $\theta(G) \le 2$.*

Proof We begin by extending G_k to a triangulation of the surface. Subdivide every nontriangular face by adding a new vertex adjacent to all vertices on the face boundary. If any new edge has length ε or more, subdivide it by adding vertices. Repeat this process until the resulting graph G_k' is a triangulation of the surface having all edges of length less than ε. G_k is a subgraph of G_k' which implies that $\theta(G) \le \theta(G')$. Thus, we may assume that G_k was a triangulation to begin with.

For each $i = 1, ..., k$, consider the "handle" in the polygon P_k with sides labelled a_i, b_i, a_i^{-1}, b_i^{-1}. Note that one point S is common to all 4k sides of P_k. Let p_i (resp., q_i) be the set of all points of P_k at distance 2ε (resp., ε) from side b_i. Note that $\varepsilon \le 1/2$ and so $2\varepsilon \le 1$. Thus, p_i and q_i are paths going from a_i to a_i^{-1} representing a non–null–homologous cycle (henceforth abbreviated "nnh–cycle") on the surface (but not necessarily in the graph). Let L_i be the closed region bounded by p_i, q_i, a_i, and a_i^{-1}, i.e., L_i is the set of all points of P_k to the left of p_i (as we travel from a_i to a_i^{-1}) and within distance ε of p_i (see Fig. 1 below).

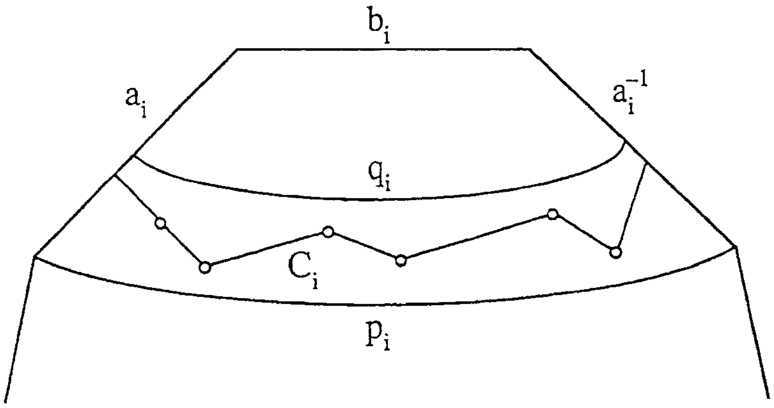

Figure 1

Within L_i, there is a path, starting at corresponding edges or vertices on a_i and a_i^{-1}, that represents a nnh–cycle in G_k: To find such a path, color a face or region of G_k blue if it meets L_i, but does not cross p_i. Color the remaining regions red. Because G_k is a 2–cell triangulation and the width of L_i is ε, we can only cross red faces as we walk along the path q_i. Similarly, we can only cross blue faces as we walk along p_i. Therefore, the boundary between the red and blue regions must lie within L_i and it must reach from a_i to a_i^{-1}, giving us a nnh–cycle in G.

For each $i = 1, ..., k$, choose such a cycle C_i, and let H_i be the subgraph of G induced by the set of all edges incident to a vertex of C_i and lying to the left of C_i. Since all vertices of C_i lie within L_i, no vertex of H_i can lie on b_i. Thus H_i is planar, since it is embedded on a cylinder. Since, by construction, the H_i's are vertex–disjoint, their union H is also planar.

Now let K be the subgraph of G induced by the edges of G – H. Then K is planar because it is embedded on the surface that remains when we cut P_k along all the nnh–cycles C_i, i.e., K is embedded on a sphere with open disks cut out and is therefore planar. Finally, $G = H \cup K$, and so $\theta(G) \leq 2$, as claimed. ❑

It follows easily from the defintions that $\theta \leq \lceil bt/2 \rceil$, where bt is the book thickness of G (also cf. [BK]). This in turn implies that $a_1 \leq 3\lceil bt/2 \rceil$. In fact, $a_1 \leq bt + 1$, as shown in the next theorem.

Theorem 6 *If G is a simple graph, then*

$$a_1 \leq bt + 1.$$

Proof The inequality clearly holds if $a_1 = 1$, so assume that $a_1 \geq 2$. By [BK Thm. 3.3], the following inequality holds:

$$bt \geq (e - n) / (n - 3), \text{ or equivalently,}$$

$$e \leq n(bt + 1) - 3bt.$$

Let H be a subgraph of G that gives the maximum value in the Nash–Williams arboricity formula. Then

$$\begin{aligned} a_1 &= \lceil e_H / (n_H - 1) \rceil \\ &\leq \lceil [n_H(bt_H + 1) - 3bt_H] / (n_H - 1) \rceil \\ &\leq \lceil bt_H + 1 + (1 - 2bt_H) / (n_H - 1) \rceil. \end{aligned}$$

Since $2 \leq a_1 = \lceil e_H / (n_H - 1) \rceil$, H is not a path and hence $bt_H \geq 1$. Thus, $1 - 2bt_H$ is negative, and therefore $a_1 \leq bt_H + 1 \leq bt + 1$. ❑

Example 1 A variation on a construction in [BK] provides an infinite class of graphs whose arboricity achieves this upper bound: For $k \geq 3$, fix $n \geq 2k + 1$ and let T_k be the triangulated n–gon shown in Figure 2.

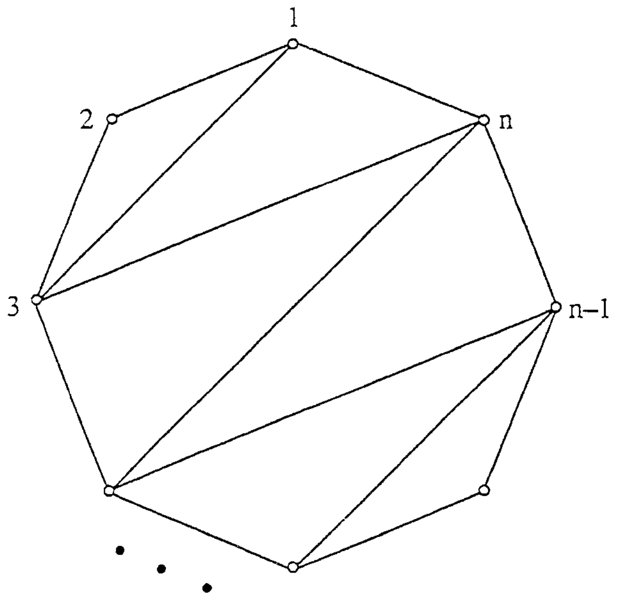

Figure 2

For $i = 0, \ldots, k-1$, let G_k^i be the graph obtained by rotating T_k through i successive positions (but leaving vertex labels fixed). Let G_k be the graph of order n obtained by taking the union of the G_k^i for $i = 0, \ldots, k-1$. Clearly, $bt(G_k) \le k$, and so $a_1(G_k) \le k+1$ by Theorem 5. Thus, it suffices to show that $a_1(G_k) \ge k+1$.

By construction of G_k, we have

$$\begin{aligned} e &= k(n-3) + n \\ &= (k+1)n - 3k. \end{aligned}$$

Next,

$$\begin{aligned} a_1 &\ge \lceil e/(n-1) \rceil \\ &= \lceil [(k+1)n - 3k]/(n-1) \rceil \\ &= \lceil k + 1 - (2k-1)/(n-1) \rceil. \end{aligned}$$

Since $n \ge 2k+1$, it follows that $a_1 \ge k+1$, and thus $a_1 = bt + 1$, as claimed.

4. Upper Bounds on Genus

It is impossible to find an upper bound for the genus γ in terms of thickness θ or arboricity a_1, as is demonstrated by the example of $K_{4,n}$, which has $\theta = 2$ and $a_1 = bt = 4$ for $n \ge 4$, but which has unbounded genus $\gamma = \lceil (n-2)/2 \rceil$ (see [BCL]). The genus of a graph G can be bounded, however, in terms of both θ and the number of vertices n (or in terms of a_1 and n, or in terms of bt and n), as shown in the following theorems. As in Theorem 3 and its corollary, the complete graph K_n shows that these results are best possible up to a constant.

Theorem 7 *If G is a simple graph with n vertices, then*

$$\gamma(G) \le (\theta(G) - 1)(n-1).$$

Proof Suppose G can be written as the union of $G_1, \ldots, G_\theta$, where the G_i's are planar and pairwise edge–disjoint. Embed each G_i on a sphere S_i; we will sew the spheres together in such a way that the resulting surface has $(\theta - 1)(n-1)$ handles.

On sphere S_1, cut out $\theta - 1$ tangents disks at each vertex i. Call the disks $D_i^{1,2}$, $D_i^{1,3}, \ldots, D_i^{1,\theta}$. On each other sphere S_k, for $k = 2, \ldots, \theta$, cut out only one tangent disk at each vertex i and call the disk D_i^k.

Join the boundaries of the cut–out disks with cylinders as follows: For $k = 2, \ldots, \theta$, attach a cylinder from $D_i^{1,k}$ (the k^{th} tangent disk to vertex i on sphere S_1) to D_i^k (the disk tangent to vertex i on sphere S_k).

The number of cylinders from S_1 to S_k equals the total number of vertices n. This is equivalent to adding $n-1$ handles to S_1. Since we do this for each value of k from 2 to θ, the resulting surface is a sphere with $(\theta - 1)(n-1)$ handles. We now erase the vertices on all spheres except S_1 and extend incident edges on S_k, $k > 1$, along the incident cylinder to the corresponding vertex on S_1. (An equivalent way to view this pasting operation is that we are simply identifying all points of the boundaries of the two disks $D_i^{1,k}$ and D_i^k, for $i = 1, ..., n$ and $k = 2, ..., \theta$.)

This gives an embedding of G on a surface of genus $(\theta - 1)(n - 1)$, and therefore, $\gamma(G) \leq (\theta(G) - 1)(n - 1)$. ❑

Since $\theta \leq \lceil bt/2 \rceil$, we obtain the following bound on γ in terms of bt and n.

Corollary 7 *If G is a simple graph of order n, then*

$$\gamma(G) \leq (\lceil bt(G) / 2 \rceil - 1)(n - 1).$$

Since $\theta \leq a_1$, Theorem 7 also implies that $\gamma(G) \leq (a_1(G) - 1)(n - 1)$. Using Euler's formula, we can cut this bound in half:

Theorem 8 *If G is a simple graph with n vertices, then*

$$\gamma(G) \leq (a_1(G) - 1)(n - 1) / 2.$$

Proof Suppose G can be written as a union of a_1 forests, each on n vertices. Then $e \leq a_1(n - 1)$. If G has a 2–cell embedding on a surface of genus γ, then Euler's formula implies

$$2 - 2\gamma = n - e + f \geq n - a_1(n - 1) + f, \text{ and so}$$
$$2\gamma \leq (a_1 - 1)(n - 1) - f + 1,$$

where f is the number of faces. Since $f \geq 1$, the result follows. ❑

Note that this result, together with the fact that $a_1 \leq bt + 1$, gives another bound on γ in terms of bt and n, almost identical to, but slightly weaker than, Corollary 7.

5. Conclusions

There are several places where the results in this work could be sharpened or improved. First, the examples known to the authors suggest that the constants can be lowered in Theorems 3, 7, and 8, and in Corollary 7. Second, it would be of interest to obtain a characterization of the graphs that satisfy $\theta = O(\gamma/n)$. Third, Theorem 5 would be improved by replacing the class of "short edge" graphs with another, more natural class of graphs of genus γ, for which a_1 and θ are bounded by a constant.

Finally, our work contains no upper bounds on book thickness. Yannakakis has proved a best possible result that $bt \leq 4$ for G planar (cf. [Y]). Heath and Istrail use that result in [HI] to show that $bt = O(\gamma)$; most recently, Malitz [M] has obtained a result of best possible order by showing that $bt = O(\sqrt{\gamma})$.

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