Unit Bar-visibility Graphs

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Abstract
A unit bar-visibility graph ("UBVG" for short) is a graph whose vertices can be represented in the plane by disjoint horizontal line segments ("bars"), all of equal length, so that two vertices are adjacent if and only if there is an unobstructed, non-degenerate, vertical band of visibility between the corresponding bars. We establish a collection of results concerning UBVGs. These include results in the following categories:

• Fundamental graph classes, including complete, complete bipartite, and hypercube graphs;
• Outerplanar graphs;
• Heredity, including subgraphs, contractions, and subdivisions;
• Triangulations and near-triangulations;
• Trees.

In particular, we show that a tree is a UBVG if and only if it is a subdivision of a caterpillar with maximum degree 3, settling in the affirmative a conjecture of Hutchinson [8]. We also show that an outerplanar, triangle-free UBVG must have an internal dual graph (the dual graph with the vertex for the external face removed) that is a subdivision of a caterpillar with maximum degree 4.

1 Introduction and Basic Properties
A bar-visibility graph or BVG is a graph whose vertices can be represented in the plane by disjoint horizontal line segments ("bars"), such that two vertices are adjacent if and only there is an unobstructed, non-degenerate, vertical band of visibility between the corresponding bars. The study of
BVGs is motivated by VLSI design, and these graphs are well understood. Wismath [16], and independently Tamassia and Tollis [13], characterized BVGs as those graphs having a planar embedding with all cutpoints on a single face, and they provided a linear-time algorithm to test for and construct BVGs.

The study of BVGs has been generalized in a number of directions, including BVGs on surfaces [2, 9, 11, 14] and visibility graphs using objects other than line segments and/or multiple visibility directions [1, 3, 4, 7, 10, 12]. In this paper we specialize to a subclass of BVGs, namely those BVGs in which all bars have equal length. Throughout we use the terminology of [5], and all graphs under consideration are simple, since all UBVGs are simple graphs.

If $G$ is a plane graph, we call the unbounded face of $G$ the external face, and the other faces are called internal. $G^*$ denotes the dual of $G$, in which the vertices are the faces of $G$, and two vertices are adjacent if and only if the corresponding faces of $G$ share an edge. The internal dual of $G$, denoted $G'_I$, is the subgraph of $G^*$ induced by the internal faces of $G$.

1.1 Definition. A unit bar-visibility graph, or UBVG, is a graph whose vertices can be represented in the plane by disjoint horizontal line segments, all of equal length, such that two vertices are adjacent if and only there is an unobstructed, non-degenerate, vertical band of visibility between the corresponding bars. A UBV layout $U$ induces, in a natural way, a plane graph $G = G(U)$.

We assume henceforth that each bar in a UBV layout has length 1 and is at a unique vertical level, usually at integer heights. We denote the height of a bar $b$ by $y(b)$, and its left and right $x$-coordinates by $x_1(b)$ and $x_2(b)$; note that $x_2(b) = x_1(b) + 1$. Two bars in a UBV layout are called collinear if a common $x$-value is shared by an endpoint of each bar; if the two bars have the same left $x$-coordinate (and hence also the same right $x$-coordinate), then they are called flush.

1.2 Proposition. If $U$ is a UBV layout, and $C_3$ is a 3-cycle in $G = G(U)$, then $C_3$ bounds an internal face in $G$.

Proof. We prove the contrapositive. Let $C_n$ be an $n$-cycle (hence $n \geq 3$) in a UBV layout, and suppose that $C_n$ does not bound an internal face. Then the interior of $C_n$ contains either an edge or a vertex. If it contains an edge connecting two vertices of $C_n$, then $n \geq 4$, since $C_3$ is a clique. If $b$ is a bar corresponding to a vertex in the interior of the embedded cycle $C_n$, then $C_n$ must contain two edges corresponding, respectively, to visibilities to the left and right of $b$. Since all bars have unit length, this again implies that $C_n$ has at least four vertices, corresponding to two bars above $b$ and two bars below $b$, two of which protrude to the left of $b$, and two of which protrude to the right of $b$. \hfill $\square$
1.3 Corollary. No triangulation of the plane with 4 or more vertices is a UBVG.

1.4 Corollary. No UBVG contains $K_4$ as a subgraph.

1.5 Corollary. The complete graph $K_n$ is a UBVG if and only if $n \leq 3$.

Another basic property of UBVGs is that any vertex of degree 4 or more is contained in cycle. This is useful in characterizing complete bipartite UBVGs and trees that are UBVGs. The latter characterization appears in Sec. 3. The straightforward proofs of the results below are omitted.

1.6 Proposition. Let $u$ and $w$ be neighbors of a vertex $v$ in a UBVG $G$. If the bars for $u$ and $w$, in a layout of $G$, protrude on the same side of $v$, then $u, v,$ and $w$ lie on a cycle.

1.7 Corollary. Any vertex in a UBVG with degree $\geq 4$ lies on a cycle.

1.8 Corollary. Any tree that is a UBVG has maximum degree 3.

1.9 Corollary. The complete bipartite graph $K_{p,q}$, where $p \leq q$, is a UBVG if and only if $p = 1$ and $q \leq 3$, or $p = q = 2$.

The next proposition, which is an important tool in subsequent proofs, establishes that any internal face has a very specific layout, as illustrated in Fig. 1.

![Figure 1: Layout of an internal face](image)

1.10 Proposition. Let $F$ be an internal face in a UBVG $G = G(U)$, and let $t_F$ and $b_F$ denote, respectively, the top and bottom bars in the UBV layout of $F$. Then there is a vertical line $L_F$ joining the interiors of $t_F$ and $b_F$, such that each other bar of $F$ is collinear with $L_F$. 
Proof. Call the bars of \( F \) not equal to \( t_F \) or \( b_F \) intermediate bars of \( F \). Let \( c \) be the uppermost intermediate bar of \( F \); then \( c \) overlaps but is not flush with \( t_F \). Let \( L_F \) be the unique vertical line that is incident with the interior of the \( t_F \) and collinear with the left or right endpoint of \( c \). The first bar below \( c \) that overlaps \( L_F \) must be the bottom bar \( b_F \). If any other intermediate bar is not collinear with \( L_F \), then it creates visibilities between bars of \( F \) lying above and below it, producing chords in the interior of \( F \), a contradiction. \( \square \)

1.11 Corollary. If \( G \) is a UBVG with no collinear bars, then \( G \) is a near-triangulation (in other words, all faces except the external one are triangles).

1.12 Corollary. An internal face \( C \) in a UBVG has at most four cut vertices. If it has four cut vertices, they must be the top and bottom bars, and two of the neighbors of these bars, one a left bar and the other a right bar.

2 Cubes, Circular Ladders, and Heredity

Propositions 1.5 and 1.9 characterize those complete and complete bipartite graphs that are UBVGs. A third class of interest is the family of hypercubes, \( Q_n \). It is easy to see that \( Q_1 \) and \( Q_2 \) are UBVGs. We prove in this section that the other planar cube, \( Q_3 \), is not. This is part of a larger result characterizing which circular ladders, that is to say, the graphs \( K_2 \times C_n \), are UBVGs, since \( Q_3 = K_2 \times C_4 \). It is left to the reader to construct a UBV layout of \( K_2 \times C_3 \).

2.1 Proposition. \( K_2 \times C_3 \) is a UBVG.

2.2 Lemma. Let \( C_n \) be an \( n \)-cycle in a UBVG, and let \( v \) be a vertex that lies in the interior of \( C_n \). If no edge of \( C_n \) is in a triangle, then at least three bars of \( C_n \) lie above \( v \) and at least three bars of \( C_n \) lie below \( v \).

Proof. Let \( v \) be a vertex in the interior of \( C_n \); it is clear that at least two bars of \( C_n \) lie above \( v \). If exactly two bars lie above \( v \), then these bars form a triangle with the vertex in the interior of \( C_n \) that has the highest bar. An analogous argument establishes that at least three bars of \( C \) lie below \( v \). \( \square \)

2.3 Corollary. Any cycle with non-empty interior in a triangle-free UBVG must have length at least 6.

2.4 Corollary. \( K_2 \times C_4 \) and \( K_2 \times C_5 \) are not UBVGs.
The next theorem completes the characterization of the circular ladders that are UBVGs: \( K_2 \times C_n \) is a circular ladder if and only if \( n = 3 \) or \( n \) is even and \( \geq 6 \).

**2.5 Theorem.** For \( n \geq 6 \), \( K_2 \times C_n \) is a UBVG if and only if \( n \) is even.

**Proof.** Let \( G = K_2 \times C_n \), where \( n \geq 6 \). Since \( G \) is 3-connected, by Whitney’s Theorem [15] it has a unique planar embedding (up to which facial cycle bounds the external face). Hence both of \( G \)'s \( n \)-cycles are facial cycles, and one of these bounds the outer face by Cor. 2.3. Call this outer \( n \)-cycle \( C_o \), and the inner \( n \)-cycle, which bounds an internal face, \( C_i \).

For necessity, suppose that \( U = U(G) \) is a UBV layout of \( G \). All the bars of \( C_i \) lie in the interior of \( C_o \). Let \( i_T \) and \( i_B \) be the top and bottom bars of \( C_i \), and let \( o_T \) and \( o_B \) be their neighbors on \( C_o \). By Lemma 2.2, there are at least three bars of \( C_o \) above \( i_T \), and at least three below \( i_B \). Exactly one of the three bars above \( i_T \) must be the neighbor \( o_T \), and the other two are the two \( C_o \)-neighbors of \( o_T \). Hence \( o_T \) lies flush with \( i_T \), with one of these two neighbors above and protruding to its left, and the other above and protruding to its right; call them \( o_1 \) and \( o_2 \) respectively. The neighbors of \( o_1 \) and \( o_2 \) on \( C_i \) must then be the uppermost left-intermediate and right-intermediate bars of \( C_i \). The layout for the three bottom bars in each of the two \( n \)-cycles is analogous; see Fig. 2.

If there is another intermediate bar, say \( i_5 \) immediately below and flush with \( i_1 \), then its \( C_o \)-neighbor \( o_5 \) must see \( i_5 \) and \( o_1 \), but not \( i_1 \), hence it is immediately below \( i_5 \) and protrudes to its left. This in turn forces another left-intermediate vertex \( i_6 \), and its mate \( o_6 \), with \( o_6 \) immediately below and flush with \( o_5 \), as illustrated in Fig. 2. Similarly, as we move down from \( i_1 \) and \( i_2 \) to any other intermediate bars besides \( i_3 \) and \( i_4 \), we see that they must occur in consecutive pairs, with their \( C_o \)-neighbors lying flush with each other between them.

For sufficiency, note that the above argument produces an explicit UBV layout for any circular ladder with \( n \geq 6 \) and even. \( \square \)

It follows from Thm. 2.5 that, given a graph \( G \) and a contraction \( H \) of \( G \), knowing whether one of \( G \) and \( H \) is a UBVG does not in general say anything about the other graph. There is a similar lack of heredity for subgraphs and subdivisions, and we state the result as a proposition whose proof is left to the reader.

**2.6 Proposition.** If \( G \) is a graph and \( H \) is a subgraph (or subdivision, or contraction) of \( G \), then it possible for either, neither, or both \( G \) and \( H \) to be UBVGs.
3 UBVGs and Trees

In this section we confirm a conjecture of Hutchinson [8] characterizing those trees that are UBVGs. The following lemma is used in proving the conjecture.

3.1 Lemma. If $S$ is a subdivision of a tree $T$, and $S$ is a UBVG, then $T$ is a UBVG.

Proof. Let $S$ be a UBVG, with $S$ a subdivision of a tree $T$, and let $v$ be a vertex of degree 2 in $S$, with neighbors $u$ and $w$. Let $S'$ be the graph obtained by deleting $v$ and adding the edge $u-w$. We show how to obtain a layout of $S'$; the process can be repeated to obtain a layout of $T$.

If the bar for $v$ is flush with either of the bars for $u$ and $w$, we just delete it to lay out $S'$, so assume that the bars for $u$ and $w$ protrude on opposite sides of $v$. First suppose that both bars lie below $v$, with $u$ protruding on the left and $w$ on the right. Because $S$ is a tree, the bars for $u$ and $w$ cannot overlap. Because $v$ has degree 2, the regions of the plane above $v$, and below $v$ between $u$ and $w$, are empty. To form the layout for $S'$, we take the region consisting of the quadrant below and to the right of $v$ together with half-plane to the right of $v$. We flip this region vertically, using $v$ as an axis, and then reposition it so that $w$ is at the former height of $v$, with $u$ visible below it, as in Fig. 3. If instead $u$ and $w$ are on opposite sides of $v$, with $w$ above, then a similar process, not requiring a flip, lays out $S'$.

A caterpillar is a tree that consists of a path (its spine) plus pendant edges. In [8] Hutchinson gave a UBV layout for any tree that is a subdivision
of a caterpillar and has maximum degree 3. We state her result and proof for completeness.

3.2 Proposition. (Hutchinson, 2000) If $T$ is a tree with maximum degree 3, and $T$ is a subdivision of a caterpillar, then $T$ is a UBVG.

Proof. The spine of $T$ is laid out as a sequence of bars alternating between heights 0 and 1 (the heights can be perturbed slightly to maintain our assumption of distinct heights). Each pendant path beginning at a vertex $v$ of the spine at height 0 (resp., height 1) is laid out as a sequence of bars flush with and below (resp., above) $v$. \hfill \Box

Hutchinson conjectured in [8] that the converse of Prop. 3.2 is true. We confirm this conjecture in Thm. 3.5. We use the following lemma, whose straightforward proof is omitted.

3.3 Lemma. If $T$ is a tree with at least four vertices of degree 3, then $T$ is a subdivision of a caterpillar if and only if $T$ does not contain a subdivision of the graph $T_0$ shown in Fig. 4.

3.4 Lemma. If a tree $T$ is a UBVG, then $T$ does not contain a subdivision of $T_0$.

Proof. Let $T$ be a UBVG. If $T$ has three or fewer vertices of degree 3, then $T$ does not contain a subdivision of $T_0$, so assume that $T$ has four or more vertices of degree 3. By Prop. 1.7, $T$ has maximum degree 3, and by Prop. 3.1, we may assume that $T$ has no vertices of degree 2. Lay out $T$ as a UBVG and let $v$ be a vertex of degree 3. By Prop. 1.6, $v$ must have a flush neighbor on one side, say above, and two neighbors below that protrude to its left and right. If the flush neighbor of $v$ has degree 3, then again by Prop. 1.6 it has two neighbors protruding to its left and right, and
these form a cycle with the protruding neighbors of \( v \). In other words, a degree-3 vertex in a tree that is a UBVG can have at most two degree-3 neighbors; hence it cannot contain a subdivision of the graph \( T_0 \).

Hutchinson’s conjecture follows immediately from the preceding results.

3.5 Theorem. A tree \( T \) is a UBVG if and only if \( T \) has maximum degree 3 and is a subdivision of a caterpillar.

The following corollary contrasts with Prop. 2.6 for arbitrary graphs.

3.6 Corollary. A tree \( T \) is a UBVG if and only if any subdivision of \( T \) is a UBVG.

4 Outerplanar UBVGs

4.1 Definition. A graph is outerplanar if it can be embedded in the plane so that every vertex lies on the external face. If \( G \) is an embedded outerplanar graph, then the internal dual of \( G \), denoted \( G_1^* \), is the dual graph \( G^* \) with the vertex corresponding to the external face deleted. It is easy to see that a 2-connected, plane graph \( G \) is outerplanar if and only if \( G_1^* \) is a tree.

4.2 Definition. Let \( G \) be a UBVG with a UBV layout that induces an outerplanar embedding of \( G \).

1. If \( f \) is an internal face of \( G \), we define the rectangle \( \text{Rec}(f) \) to be the smallest rectangle containing all the bars of \( f \). It follows from Prop. 1.10 that \( \text{Rec}(f) \) has width \( > 1 \) and \( \leq 2 \), and that the top and bottom bars of \( f \) lie on the upper and lower borders, resp., of \( \text{Rec}(f) \).

2. Let \( f \) and \( f' \) be internal faces of \( G \), and let \( f' \) be a neighbor of \( f \) in \( G_1^* \) (i.e., the two faces share an edge). If \( \text{Rec}(f') \) protrudes to the left
of $\text{Rec}(f)$, we call $f'$ a left neighbor of $f$. It is not hard to see that a left neighbor of $f$ cannot simultaneously be its right neighbor.

3. If $f'$ is a neighbor of $f$, but neither a left nor right neighbor, then $f'$ contains bars above $\text{Rec}(f)$ or below $\text{Rec}(f)$, but not both, since $G$ is outerplanar. In the former case we call $f'$ a top neighbor of $f$, and in the latter case we call it a bottom neighbor of $f$. In both cases $\text{Rec}(f')$ is either left-flush or right-flush with $\text{Rec}(f)$, and we call $f'$ a left-flush or right-flush neighbor accordingly.

The following results give necessary conditions for an outerplanar graph to be a UBVG.

4.3 Proposition. Let $f$ be an internal face in an outerplanar UBVG $G$. Then $f$ has at most one left and one right neighbor, and at most two top and two bottom neighbors. If $f$ has two top (resp., bottom) neighbors, then both are triangles.

Proof. If $f_1$ and $f_2$ are distinct left neighbors of $f$, then bars from each face protrude to the left of $\text{Rec}(f)$, creating a cycle in $G_1^*$ – impossible since $G$ is outerplanar. Hence $f$ has at most one left neighbor, and likewise at most one right neighbor. A top (resp., bottom) neighbor of $f$ is incident with an edge joining the top bar of $f$ to an uppermost intermediate bar of $f$. By Prop. 1.10 $f$ has at most two such edges, and so at most two top (resp. bottom) neighbors.

Now suppose $f$ has two top neighbors, $f_L$, is left-flush and $f_R$, right-flush. Thus $f_1$ has no left-intermediate bars, and $f_R$ has no right-intermediate bars. Furthermore, the top bar of $f$ is a right-intermediate bar for $f_L$ and a left-intermediate bar for $f_R$. If either $f_L$ or $f_R$ has any other intermediate bars, then $G$ is not outerplanar. Therefore $f_L$ and $f_R$ are both triangles. □

4.4 Corollary. If $G$ is a 2-connected, outerplanar UBVG, then the maximum degree in $G_1^*$ (i.e., the maximum length of an internal face) is 6. If, in addition, $G$ is triangle-free, the maximum degree in $G_1^*$ is 4, and each internal face has at most one left neighbor, one right neighbor, one top neighbor, and one bottom neighbor.

The next result gives a simple, necessary condition for a 2-connected outerplanar, triangle-free graph to be a UBVG.

4.5 Theorem. If $G$ is a 2-connected, outerplanar, triangle-free UBVG, then $G_1^*$ is a subdivision of a caterpillar with maximum degree 4.

Proof. Suppose not. By Cor. 4.4, $G_1^*$ has maximum degree 4, and by Lemma 3.3, $G_1^*$ contains a subdivision of the graph $T_0$ shown in Fig. 4. Let $S_0^*$ be a subgraph of $G_1^*$ that is a subdivision of $T_0$, with vertex $f$
corresponding to the center degree-3 vertex of $T_0$, and vertices $a, b, c$ corresponding to the other degree-3 vertices of $T_0$. There are three vertex-disjoint paths $P_i, i = a, b, c$ from $f$ to $a, b, c$ respectively. Furthermore, there are six distinct vertices $i_1$ and $i_2$, for $i = a, b, c$, which are neighbors of vertex $i$, and which do not lie on any of the paths $P_i$. Let $x_i$ on $P_i$ be the last vertex after $f$ that is a top or bottom neighbor of its predecessor on $P_i$. If $x_i \neq i$, then the next vertex on $P_i$ is a left or right neighbor of $x_i$, by Cor. 4.4. If $x_i = i$, then either $i_1$ or $i_2$ is a left or right neighbor of $x_i$. In either case denote this left or right neighbor of $x_i$ by $f_i$, and the path from $f$ to $f_i$ by $Q_i$. Without loss of generality, $f_1$ and $f_2$ are both left neighbors of their predecessor on $Q_i$ (see Fig. 5, which uses arrows to indicate whether a face is a top, bottom, or left neighbor of the preceding face on its path). Thus one of the bars from $f_1$ or $f_2$ must see a bar from a face on $Q_1 \cup Q_2$ that is not its neighbor, implying that $G_f^*$ is not a tree – a contradiction since $G$ is an outerplanar UBVG.

\[\text{Figure 5: Subgraph of } G_f^* \text{ in proof of Thm. 4.5}\]

4.6 Definition. Let $T$ be a subdivision of a caterpillar with maximum degree 4, embedded in the plane. If $T$ has a spine $S$ with the property that any vertex with degree 4 has one of its two non-spine neighbors on one side of $S$ and the other on the other side of $S$, we’ll say that $T$ has a double comb embedding. If $T$ is embedded as a double comb, we orient the edges $T$ from left to right along the ‘spine’ of the comb, low to high along ‘upper teeth,’ and high to low along ‘lower teeth.’

4.7 Corollary. If $G$ is a 2-connected, triangle-free, outerplanar UBVG, then the graph $G_f^*$ induced by a UBV layout of $G$ is a double comb.

Proof. For the spine $S^*$ in $G_f^*$ we choose a longest left-right path of faces in $G_f^*$.
The results that follow give sufficient conditions for a graph with an outerplanar embedding to be a UBVG.

### 4.8 Proposition

If $G$ is a 2-connected, outerplanar embedded graph such that $G^*_r$ is a path, then $G$ is a UBVG.

**Proof.** Suppose $G^*_r$ is the path $P^* = f_1 - f_2 - \ldots - f_k$, thought of as being numbered from left to right within the outer face; give the outer face label $f_{k+1}$. Choose a degree-2 vertex in face $f_k$, label it $v_1$, and then label the remaining vertices consecutively, clockwise around the outer face, as in Fig. 6. We lay out the bar for vertex $i$ at height $i$, $i = 1, \ldots, n$. The bars 1 and $n$ extend horizontally from $x$-coordinates $k + 1$ to $2(k + 1)$, and the bar for each other vertex $v$ extends horizontally from $m$ to $2m$, where $f_m$ is the highest numbered internal face containing $v$. 

![Figure 6: Layout of outerplanar UBVG when $G^*_r$ is a path](image)

The last result of this paper gives sufficient conditions for a 2-connected, outerplanar graph $G$, with internal dual $G^*_r$ a double comb, to be a UBVG. Its proof uses the span of a UBVG and the characterization of UBVGs with spans 1 and 2 as, respectively, paths and zigzag ladders (defined below); both these types of graphs are outerplanar.

### 4.9 Definition

If $U$ is a UBV layout, then $\text{Span}(U)$ is the number of distinct left endpoints of the bars of $U$. If a graph $G$ is a UBVG, then $\text{Span}(G)$ is the minimum value of $\text{Span}(U)$ such that $G \cong G(U)$.

### 4.10 Proposition

A graph $G$ is a UBVG with span 1 if and only if $G$ is a path.

### 4.11 Definition

A zigzag ladder is a connected graph $G$ that can be drawn in the plane with all its vertices lying on two vertical paths, so that if the edges of these two paths are directed upward, then the remaining edges of $G$ can be directed to yield a directed Hamiltonian path in $G$. An example is given in Figure 7.
4.12 Proposition. A connected UBVG $G$ has span 2 if and only if $G$ is a zigzag ladder.

4.13 Theorem. Let $G$ be a 2-connected outerplanar embedded graph, such that $G^*_1$ has a longest possible spine $S^*$ making it a double comb, and such that any face $f$ having degree 3 or 4 in $G^*_1$ has the following properties:

1. If $f$ has two neighbors sharing edges $e_1$ and $e_2$ with $f$ respectively, then the endpoints of $e_1$ are distinct from the endpoints of $e_2$;

2. If $f$ has left and right neighbors sharing edges $e_L$ and $e_R$ with $f$ respectively, then the endpoints of $e_L$ are not adjacent to the endpoints of $e_R$;

3. The left and right neighbors of $f$ (which exist by the maximality of $S^*$) are both 4-cycles in $G$, all of whose vertices have degree at most 3;

4. If $P = f_1 - f_2 - \ldots - f_m$ is an oriented path of faces comprising a “tooth” of the double-comb $G^*_1$, with $f_1$ a top or bottom neighbor of $f$, then $P$ is a zigzag ladder.

Then $G$ is a UBVG.

Proof. We lay out each face $f$ having degree 3 or 4 in $G^*_1$ so that its edge $e_L = u_L - v_L$ (resp., $e_R = u_R - v_R$) has $u_L$ and $v_L$ laid out as left-intermediate bars (resp., $u_R$ and $v_R$ laid out as right-intermediate bars). These then serve as the top and bottom bars for the neighboring 4-cycle specified in property 3, whose remaining two bars are placed as left-intermediate bars between $u_L$ and $v_L$ (resp., right-intermediate bars between $u_R$ and $v_R$), hence protruding to the left (resp., right) of $f$. See Fig. 8.

If $f$ has a top neighbor in $G^*_1$, which by assumption is part of a zigzag ladder $L_T$ comprising a “tooth” of the double comb $S^*$, we choose the
second vertex in the directed Hamiltonian path $H$ of $L_T$ to represent the top bar $t_f$ in the UBV layout of $f$ (since $H$ shares its first edge with $f$, this vertex is on $f$). Property 1 implies that this vertex is on neither $e_L$ nor $e_R$, and property 4 guarantees that $L_T$ can be laid out without protruding to the left or right of the edge it shares with $f$. If $f$ does not have a top neighbor we choose a vertex of $f$ lying on the clockwise path from the right vertex of $e_L$ to the left vertex of $e_R$, but belonging to neither edge, which is guaranteed by property 2. We choose the bottom bar $b_f$ in an analogous manner. Thus the bars of $e_L$ (resp., $e_R$) are left bars (resp., right bars) of the layout for $f$.

Next let $P$ be a maximal subpath in $S^*$ consisting of faces with degree 1 or 2 in $G^*_T$, but excluding left or right neighbors of faces with degree 3 or 4. If $f'$ is one of the two end-faces on $P$, then $f'$ might be the left (resp., right) neighbor of a 4-cycle $C$ that is itself the left (resp., right) neighbor of a face $f$ having degree 3 or 4 in $G^*_T$. Using the proof of Prop. 4.8, we can lay out $P$ in such a way that the edge $e$ shared by $f'$ and $f$ has its vertices flush and protruding to the left (resp., right) of the rest of the layout of $P$ if $f'$ is a right (resp., left) neighbor of $f$. These two bars can then be placed between the other two bars of the 4-cycle, whose layout was done together as part of the layout of $f$. By repeating this process for each such subpath $P$ of $S^*$, we lay out the entire graph $G$.

\[\square\]

5 Conclusions and Acknowledgments

This paper provides a number of fundamental results concerning UBVGs, but many unanswered questions remain. Of particular interest to the authors is a complete characterization of outerplanar UBVGs, and a characterization of all UBVGs with a given span. A larger open question is whether a simple characterization of all UBVGs, perhaps analogous to BVG characterizations of [13, 16], is possible.
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