DOING MATHEMATICS — A SKIDMORE VIEW

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A mathematician is a machine that turns coffee into theorems.

"This work has applications in telephone network design, but that's not why I'm interested in it."

Half a joke: The President of Skidrow College announced at a meeting of department chairs that, due to a fiscal crisis, big-budget departments would have to be eliminated. The chair of the mathematics department replied, "Well, we're safe then. All we need for our research are paper, pencils, and a wastebasket."[1]

What is that makes mathematicians do what they do — what exactly is it that they *do* do — and how and why do they do it? Mathematicians are regarded with suspicion and fear by most of the rest of the human population. When I introduce myself to someone and say I'm a mathematician, the response is likely to be, "Oh God, I'm awful at math," or, "Could you help me with my taxes?" If I were an artist, I expect the response would be something else entirely. Perhaps I'd be asked what type of art I do, or maybe I'd be asked to show some of my work. Yet the mathematician and artist are not so different, and our respective creations are both conceived with varying combinations of inspiration and toil. I've never tried to write *about* doing mathematics before, although I've written a fair amount of mathematics. Nonetheless, I'll try here to step back and examine what it is that I and other mathematicians do when we do mathematics.

First of all, what is it that mathematicians do? The world is full of interesting connections and symmetries that can be expressed numerically, algebraically, or geometrically; it's full of puzzles and paradoxes that draw mathematicians in and ask to be solved. Mathematicians try to find interesting connections and symmetries and prove that they hold (or that they don't); we try to solve puzzles that are posed to us or that we pose ourselves. Some of these are well-known: most educated people have heard of the Pythagorean Theorem (sixth century B.C.), which finds a relationship among the lengths of the sides of any triangle that has a right angle, namely $a^2 + b^2 = c^2$, where a, b, and c are the lengths of the sides, with c the length of the side opposite the right angle. Others are not so well known outside the mathematical community: In 1874 Georg Cantor proved that some infinite sets are "more infinite" than other infinite sets (in a very precise sense!); in 1936 Alan Turing proved that there are some computer problems that can

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[1] To complete the joke, put your least favorite department X in the following: The chair of department X then stood up and said, "We can do even better than that — for our research, we don't need the wastebasket!"

never be solved by a computer program. Each century only a handful of theorems will appear that are amazing, beautiful, or important enough to be known by the general mathematical community or even beyond it. Nonetheless, countless mathematicians have, do, and will toil away on the problems that they're driven to solve.

How do mathematicians find problems to work on? Some problems have been around for years or centuries. In 1637, Pierre de Fermat wrote in the margin of a textbook about a question involving solutions to equations, "I have assuredly found an admirable proof of this, but the margin is too narrow to contain it." In 1994, over 350 years later, this problem (known as "Fermat's Last Theorem") appears to have finally been solved by a mathematician named Andrew Wiles — perhaps now it will become known as "Wiles' Theorem." A host of other "vintage" problems remains stubbornly unsolved. Many of the problems we work on come from reading one another's work, or are inspired by "real-life" problems that are brought to us by colleagues in other disciplines. These problems range widely in topic and difficulty.

A recent example from my own work was inspired by a paper in an engineering journal that studies the design of printed electrical circuits. Like the anonymous mathematician quoted at the beginning of this article, I'm not particularly excited about the actual electrical circuits, but they generate mathematical puzzles that fascinate me and various other mathematicians who also study this topic. I actually heard about this problem not from the paper, but from a mathematical colleague who took me aside at a conference and sketched out the problem on a piece of scrap paper. In order to give you a sense of the process a mathematician might go through in solving a problem, I'll tell you a little about my experience working on this question.

The circuits in the article had a special type of design. They consisted of wires connecting a collection of nodes, and these networks of wires and nodes were to be "painted" onto a printed circuit board. The nodes were to be put on as solid rectangles, and the wires were to be horizontal and vertical lines connecting the nodes. Here's a picture of such a layout:

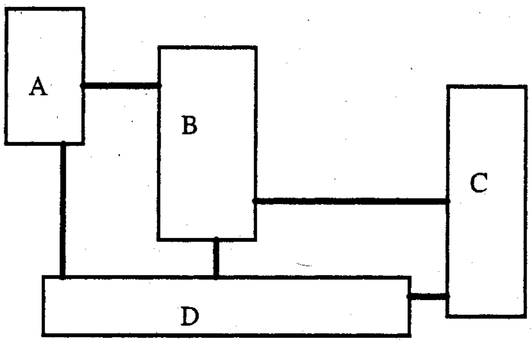

Figure 1. A rectangle-visibility layout

Any such diagram is called a *rectangle-visibility layout* of the corresponding electrical circuit. Notice in this layout that we can't draw a wire from rectangle A to rectangle C without crossing rectangle B; we say that C is not *visible* from A. Of course, if we drew the diagram a little differently, say by making C taller, then we wouldn't have that problem. This motivates the following mathematical problem: Given a particular circuit, in which certain nodes should be connected by wires and others should not, how can we design a rectangle visibility layout to represent it — indeed, is such a layout possible? For example, we see that by altering the example above, we could design a layout to represent a circuit with 4 nodes, all of which are connected to each other. Could we do this for a circuit with 5 nodes, 6 nodes, *any* number of nodes, all of which are connected to one another?[2]

Joan Hutchinson of Macalester College is a colleague with whom I do a lot of my research. She and I set out to answer this type of question for a certain class of circuits that I'll call *bipartite* (meaning two parts). In bipartite circuits, the nodes are divided into two groups, with all the nodes in one group to be connected to all the nodes in the other group, but *no* connection should be possible for two nodes in the same group. For instance, here's a rectangle-visibility layout for a circuit with 3 nodes in one group (labeled A, B, C) and 4 nodes in the other (labeled 1, 2, 3, 4). I haven't drawn in the wires, but you can check that the circuit I drew really is bipartite. This circuit is called $K_{3,4}$ for short.

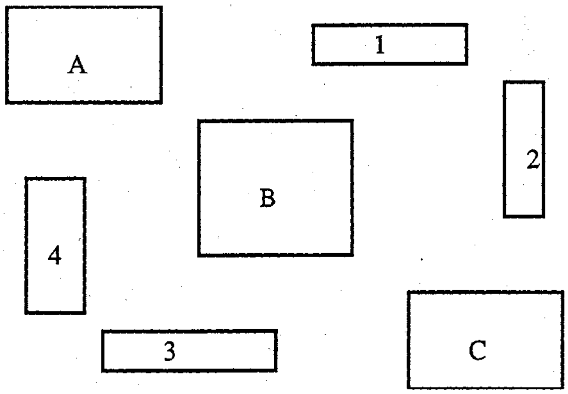

Figure 2. A rectangle-visibility layout of $K_{3,4}$

Joan and I became interested in the question of *which* bipartite circuits could be drawn this way. Working with other colleagues on math problems is for me a wonderful part of being a mathematician. Even though Joan lives over a thousand miles away in Minneapolis, we're able to keep in touch by e-mail notes that we send back and forth. Once or twice a year, we'll get together, usually at math conferences, where we can talk at length with each other and share ideas with other mathematicians. Mathematics dominates the conversation as we hike up a mountain, or as we indulge our common liking for micro-brewery beer. Diagrams and ideas are sketched out in the air with fingers or on napkins at restaurants. Then we go back home, trying out new approaches, looking at examples for new ideas, and go back to our e-mail notes for feedback.

While many mathematicians thrive on this type of camaraderie, some researchers prefer to work alone. I've heard it said that Andrew Wiles (of Fermat's Last Theorem fame) is the latter type of mathematician. At the other end of the spectrum is one of the most famous living mathematicians, Paul Erdös, who is now over 80 years old. Erdös so loves traveling and working with other

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[2] It turns out that the answers to these questions are *yes*, *yes*, and *no*. You might want to try drawing some yourself and see how many nodes you can draw layouts for.

mathematicians that he has no permanent residence. He has written more than 1000 papers and has had more than 250 co-authors; to stimulate his colleagues to work on problems he finds interesting, he offers cash prizes from $10 to $10,000. His legendary number of co-authors has led his mathematical friends to define a whimsical quantity called the "Erdös number." Each mathematician gets an Erdös number as follows: Erdös gets the number 0; if you've ever co-authored with Erdös you get the number 1; if you've ever co-authored with a co-author of Erdös you get the number 2, etc. I believe my Erdös number is 3.

In any case, alone or with friends, mathematicians work away at the problems that interest them, a process that can take days, months, or years. As Joan and I thought about our circuit question, we tried to make examples of rectangle-visibility layouts for specific bipartite circuits, like $K_{3,4}$ shown in Figure 2. Working with examples gives us a better feeling for what the general answer might be. Pretty quickly we were able to construct examples for any bipartite circuit in which at least one of the two groups had at most four nodes. However, try as we might, we could *not* find a layout for any circuit in which *both* groups had *five* or more nodes; in particular, neither of us could find a layout for $K_{5,5}$, in which each of five nodes in one group must be visible to each of five nodes in another group, but with no visibilities within either group. We found a layout missing only one visibility, and we found another layout with only one extra, unwanted, visibility, but we couldn't find one that was just right!

The evidence provided by examples (or by the inability to find examples) leads mathematicians to make *conjectures* — educated guesses — about what might be true in general. At this point in our work, Joan and I conjectured that it was *impossible* to draw a rectangle-visibility layout for $K_{5,5}$. Now the hard work began. Over a period of months, we each worked on this question, sometimes trying to find a layout for $K_{5,5}$, sometimes trying to show that none could exist. I do most of my work in a spiral-bound notebook, because ideas that I think are worthless one day may turn out later to contain some useful nugget of insight that I hadn't noticed earlier. My notebook contains my own mathematical fiddling, notes I write down from books or papers that I think might help, as well as personal exclamations of frustration or satisfaction. At one point in the notes from that time, I wrote, "So I have *something* here, but what is it?" Close to year later I wrote, "Aargh — I'm spinning my wheels."

Mathematics research requires patience, diligence, and (hopefully) occasional flashes of insight. Most people work on two or more problems at once, so that they can have a change of pace if they're not getting anywhere on one of them. As you leave problems for a while and then come back to them, you hope you'll find a new way of looking at them that will reveal the solution. For the $K_{5,5}$ problem, on which Joan and I worked for many months, my key flash of insight came one night as I drove home with my family from a day at Disneyland (we were spending the year in San Diego on sabbatical). Joan and I had believed for some time that the problem for $K_{5,5}$ was related to the fact that there were only four visibility directions (up, down, left, and right), but five rectangles needing to see five other rectangles. As my car languished in an L.A. traffic jam in the dark, my mind wandered to this question, and I got an idea about how to get at the problem. It used a simple but powerful principle, called the "Pigeonhole Principle,"[3] in a slightly sneaky way.

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[3] The Pigeonhole Principle says: If a certain number of objects are to be placed in a certain number of boxes (or pigeonholes), and if there are more objects then there are boxes, then some box will contain more than one object.

It also involved expressing part of the problem in a different manner, using a table with five rows and five columns to represent a layout. Finding a different way to express your question is often a key step in solving mathematical problems; each different expression reveals different aspects of the situation. By doing this, I was able to show the following. *If* there were a rectangle-visibility layout for $K_{5,5}$, part of it would have to look like the following (where I use letters for the rectangles in one group, and numbers for the rectangles in the other group):

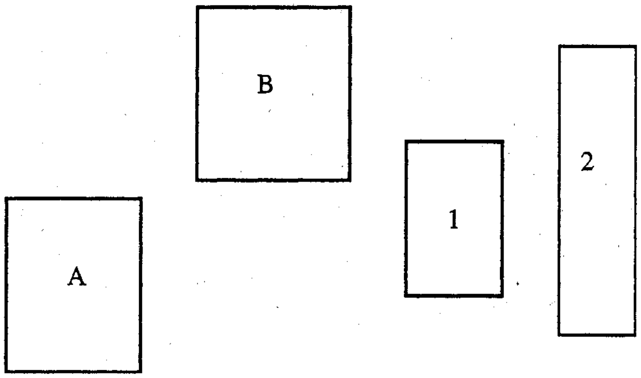

Figure 3. Partial rectangle-visibility layout of $K_{5,5}$

Now this is not the complete picture, of course — there have to be ten rectangles all together. But this is a starting point. For instance, there definitely has to be a letter rectangle between rectangles 1 and 2, because they're not allowed to be visible to each other. Continuing in this way, Joan and I were able to say where each of the remaining rectangles would have to go, until in our last step we were able to show that there was *no* place for the tenth rectangle to go. This finally proved our conjecture: there is *no* rectangle-visibility layout for $K_{5,5}$.

The feeling of elation when you solve a problem that's been on your mind for weeks or months is fantastic. It's more than just the feeling of success after working hard on something for a long time. There's a great feeling of satisfaction at putting in its proper place one more piece of the puzzle — albeit a small one — that comprises the complex but orderly mathematical structure of our world. Most mathematicians, myself included, will not ever prove a result that will be written about in the New York Times, or even become known beyond the relatively small group of mathematicians who work on similar questions. Nonetheless, we are part of a community of scholars who find the same beauty in the revelation of a previously unknown connection between two ideas as artists do in an inspired painting, or as astronomers do when they find a new understanding of the universe, or as poets do when they put words together in a new and profound way. I remember in college that it knocked my socks off to think about different levels of infinity, as Cantor did in the 19th century, and I still get that same thrill today when I hear a great mathematician describe some major new and exciting result, as well as when I discover my own, perhaps not so major, new results. I hope I am able to convey some of the joy of discovery to my students, but in the end it's something you can really only experience by doing it yourself.