When is an Algebraic Duality a Geometric Duality?*

Alice M. Dean  
Department of Mathematics and  
Computer Science  
Skidmore College

R. Bruce Richter  
Department of  
Mathematics and Statistics  
Carleton University

Abstract

In this note, we give a simple characterization of when an algebraic duality is attained as the geometric duality of a planar embedding.

1. Introduction

An algebraic duality is a bijection $f : E(G) \rightarrow E(G')$ from the edges of a graph $G$ to the edges of a graph $G'$ such that $p$ is the edge-set of a polygon of $G$ if and only if $f(p)$ is the edge-set of a minimal edge-cut of $G'$. (Throughout this paper, a polygon is a connected 2-regular graph and an edge-cut is a set $c$ of edges for which there is a partition $(X,Y)$ of the vertices so that the edges with one end in $X$ and one end in $Y$ are precisely the edges in $c$. It is minimal if it does not properly contain any other edge-cut.)

A geometric duality is a bijection $f : E(G) \rightarrow E(G')$ from the edges of a graph $G$ to the edges of a graph $G'$ such that there is a planar embedding of $G$ whose dual is $G'$ and such that (in the usual planar duality) the edge $e$ of $G$ is dual to the edge $f(e)$. It is well-known (and straightforward to show) that every geometric duality is an algebraic duality. A famous theorem of Whitney [7] says that if $G$ is 2-connected and if $f$ is an algebraic duality, then $f$ is also a geometric duality.

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It has long been known that if \( G \) is not 2-connected, it can have an algebraic duality that is not a geometric duality. (A simple algebraically self-dual example is shown in Figure 1.) The main result of this note is a simple necessary and sufficient condition for an algebraic duality \( f : E(G) \to E(G') \) to be a geometric duality.

Figure 1

The study of algebraic duality was motivated in part by recent work on self-dual planar graphs [1,2,3,4,5,6]. The article [1] provides a set of constructions that produce all self-dual planar maps. This same thing is done by rather different techniques in [6], where, more completely, all pairs \((G, \phi)\) of self-dual maps \( G \) and map isomorphisms \( \phi \) between \( G \) and its geometric dual are produced. We note that it is straightforward to construct all algebraically self-dual graphs — these are the graphs whose blocks come in dual pairs (where a self-dual block can be both elements of the pair).

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2. Elementary Properties of Algebraic Duals

In this section, we develop some of the basic theory of algebraic duality. Much of this is already known, but is provided here for completeness.

We use the notation \( G - e \) to denote \( G \) with the edge \( e \) deleted and the notation \( G/e \) for \( G \) with the edge \( e \) contracted.

**Lemma 1.** Let \( G \) and \( G' \) be graphs and let \( f : E(G) \to E(G') \) be an algebraic duality from \( G \) to \( G' \). Then:

1. if \( I \) and \( I' \) are the sets of isolated vertices of \( G \) and \( G' \), respectively, then \( f : E(G - I) \to E(G' - I') \) is an algebraic duality;
2. for each \( e \in E(G) \), both the restrictions \( f : E(G) \setminus \{e\} \to E(G' \setminus f(e)) \) and \( f : E(G/e) \to E(G'/f(e)) \) are algebraic dualities.
3. the function \( f^{-1} : E(G') \to E(G) \) is an algebraic duality; and
4. if \( B \) is any block of \( G \), then \( f(E(B)) \) induces a block of \( G' \).
Proof. (1) is obvious. (2) is easily checked directly, although there are a few cases to consider.

(3) We proceed by induction on \( \varepsilon = |E(G)| \). For \( \varepsilon = 1 \), the only algebraic dual of the link is the loop and the only algebraic dual of the loop is the link, so the result holds in this case. Thus, we can assume \( \varepsilon > 1 \).

We must show that \( f(e) \) is a polygon of \( G' \) if and only if \( e \) is a minimal edge cut of \( G \). This is easily done by considering the results of contracting and deleting. For example, suppose \( e \) is a minimal edge cut of \( G \). If \( e \) has only one edge, then we contract all the edges of \( G - e \) and apply the base case and (2) to conclude that \( f(e) \) is a loop. If \( e \) has two or more edges, then \( e \setminus e \) is a minimal edge cut of \( G - e \), so, inductively, \( f(e \setminus e) \) is a polygon of \( G'/f(e) \). Since \( e \setminus e \) is not a minimal edge cut of \( G/e \), \( f(e \setminus e) \) is not a polygon of \( G' - f(e) \). Therefore, \( f(e) \) is a polygon of \( G' \), as required.

(4) We contract every edge of \( G \) not in \( B \) to find that \( f(E(B)) \) induces a subgraph of \( G' \) that is an algebraic dual of \( B \). By Whitney's Theorem, this is 2-connected. Therefore, \( f(E(B)) \) induces a 2-connected subgraph of some block \( B' \) of \( G' \). Similarly, (3) implies that \( f^{-1}(E(B')) \) is contained in some block of \( G \). This implies that \( B' \) is the subgraph of \( G' \) induced by \( f(E(B)) \), as required.

3. The Connected Case

In this section, we consider the case that \( G \) is connected and provide, for such \( G \), a characterization of when an algebraic duality is attained as a geometric duality. But first we need a little notation.

A cutpoint of a graph \( H \) is a vertex \( v \) such that there exist two edge-disjoint subgraphs \( K, K' \) of \( H \), each having at least one edge, so that \( K \cup K' = H \) and \( K \cap K' \) is just the isolated vertex \( v \).

For a vertex \( v \) of a graph \( H \), \( \delta(v) \) is the set of edges incident with \( v \). To help keep things clear, we shall denote vertices and edges of \( G \) by \( v, w, \ldots \) and \( e, \ldots \), and we shall denote vertices and edges of an algebraic dual \( G' \) of \( G \) by \( v', w', \ldots \) and \( e', \ldots \).

**Theorem 2.** Let \( G \) and \( G' \) be connected graphs and let \( f : E(G) \to E(G') \) be an algebraic duality. Then \( f \) is attained as a geometric duality if and only if, for each cutpoint \( v' \) of \( G' \), the subgraph of \( G \) induced by \( f^{-1}(\delta(v')) \) is connected.
As an example, let $f$ be the algebraic self-duality for the graph in Figure 1 that interchanges the loop and the isthmus and maps each of the other two edges onto itself. If $v'$ is the middle vertex of the three, then $f^{-1}(\delta(v'))$ is the loop and the two parallel edges, which do not induce a connected subgraph. Therefore, $f$ cannot be a geometric duality.

**Proof.** *Necessity.* Let $G$ be embedded in the plane so that $f$ is attained as the geometric duality of the embedding. Suppose $v'$ is a cutpoint of $G'$. There is a face $F$ of the embedding of $G$ corresponding to $v'$ and $f^{-1}(\delta(v'))$ consists of the edges in the boundary of $F$. Since $G$ is connected, the boundary of $F$ is connected.

*Sufficiency.* The proof is by induction on the number of blocks of $G$. If there is only one, then the result follows from the well-known result of Whitney [7].

Let $v'$ be a cutpoint of $G'$ and let $B'_1, B'_2, \ldots, B'_k$ be the blocks of $G'$ containing $v'$. Let $\delta_i = \delta(v') \cap E(B'_i)$, for $i = 1, 2, \ldots, k$. For those $\delta_i$ which are not just a singleton loop, $\delta_i$ is a minimal edge-cut of $G'$ and so $P_i = f^{-1}(\delta_i)$ is the edge-set of a polygon of $G$. If $\delta_i$ is just a loop, then $P_i = f^{-1}(\delta_i)$ is an isthmus of $G$ (since $f^{-1}$ preserves blocks, by Lemma 1 (3) and (4)). (We note that we use $P_i$ to denote both an edge-set and the subgraph induced by the edge-set.)

By hypothesis, the subgraph $H = \bigcup_{i=1}^k P_i$ of $G$ is connected. Because $f$ and $f^{-1}$ preserve edge-sets of blocks, each $P_i$ is in a different block of $G$, so is a block of $H$. Thus, one of them, say $P_j$, contains only one cutpoint of $H$.

Let $G'_1$ be the maximal connected union of blocks of $G'$ containing $\bigcup_{i \neq j} B'_i$ but not containing $B'_j$. Let $G'_2$ be the union of the remaining blocks of $G'$, so $B'_j \subseteq G'_2$. For $i = 1, 2$, let $G_i$ denote the subgraph of $G$ induced by the edges in $f^{-1}(E(G'_i))$.

From the hypothesis and the fact that $f^{-1}$ preserves blocks, it is clear that $G_2$ is connected. The connection of $G_1$ follows from the same two facts plus the fact that $\bigcup_{i \neq j} P_i$ is connected. (This was why we chose $P_j$ to have only one cutpoint of $H$.)

In the following paragraphs, we shall make use of the following observation: if a connected graph $H$ has a cutpoint $v$ and subgraphs $K$ and $K'$ such that $K \cup K' = H$ and $K \cap K'$ is just $v$, then every minimal edge-cut of either $K$ or $K'$ is a minimal edge-cut of $H$ and every minimal edge-cut of $H$ is a minimal edge-cut of either $K$ or $K'$.
We claim that, for $i = 1, 2$, $f : E(G_i) \rightarrow E(G'_i)$ is an algebraic duality. For if $P$ is the edge-set of a polygon of $G_i$, then $P'$ is the edge-set of a polygon of $G$, so $f(P)$ is a minimal edge-cut of $G'$. By the preceding comment, $f(P)$ is a minimal edge-cut of either $G'_i$ or $G'_i$; obviously it is the former, as required.

For the other direction, if $c'$ is a minimal edge-cut of $G'_i$, then, by the earlier observation, $c'$ is a minimal edge-cut of $G'_i$. Therefore, there is a polygon of $G$ whose edge-set $P$ satisfies $f(P) = c'$. Obviously, $P$ is also a polygon of $G_i$.

In order to apply the inductive assumption, we need to show that if $w'$ is a cutpoint of $G'_i$, then $f^{-1}(\delta(w'))$ induces a connected subgraph of $G_i$. But $G'_i$ is the union of blocks of $G'$, so that $w'$ is also a cutpoint of $G'$ and, therefore, $f^{-1}(\delta(w'))$ induces a connected subgraph of $G_i$ and, therefore, of $G_i$.

By the inductive assumption, there is an embedding of $G_i$ in the plane such that $f : E(G_i) \rightarrow E(G'_i)$ is attained as the geometric duality. If follows that there is such an embedding in which $c'$ corresponds to the infinite face of $G_i$. Let $v$ be the vertex of $G$ common to $P_j$ and $\bigcup_{i \neq j} P_i$. In both the embeddings, $v$ is on the infinite face, so they can be joined by identifying the two copies of $v$ to yield an embedding of $G$ with geometric dual $G'$ and $f : E(G) \rightarrow E(G')$ is attained as the geometric duality of the embedding.

### 4. The Disconnected Case

In this section, we consider the situation in which $G$ is not connected. We can use Theorem 2 as a base to prove the following.

**Theorem 3.** Let $G$ have no isolated vertices and let $f : E(G) \rightarrow E(G')$ be an algebraic duality. For each component $K$ of $G$, let $K'$ denote the subgraph of $G'$ induced by $f(E(K))$. Then $f$ is a geometric duality if and only if: (1) $G'$ is connected; (2) each component $K$ of $G$ is connected; and (3) for each component $K$ of $G$, and each cutpoint $v'$ of $K'$, $f^{-1}(\delta(v')) \cap E(K)$ induces a connected subgraph of $K$.

**Proof.** Necessity. Let $G$ be embedded in the plane so that $f$ is attained as the geometric duality of the embedding. Then (1) and (2) are obvious. For (3), let $K$ be a component of $G$. Delete all the edges in $E(G) \setminus E(K)$ (and delete the resulting isolated vertices).
Dually, we contract all the edges in \( E(G') \setminus E(K') \), which leaves \( K' \). By Lemma 1, \( f : E(K) \to E(K') \) is an algebraic duality, and this is attained as a geometric duality, so Theorem 2 applies, yielding (3).

**Sufficiency.** The result is proved by induction on the number of components of \( G \). In the case \( G \) is connected, the result follows from Theorem 2. So we can suppose \( G \) is not connected.

Let \( T^* \) be a new bipartite graph, whose vertices are the components of \( G \) and the cutpoints of \( G' \), with the component \( K \) being joined to the cutpoint \( v' \) by an edge of \( T^* \) if \( v' \) is a vertex of \( K' \). It is clear that \( T^* \) is a tree. Delete from \( T^* \) any cutpoint vertices that are adjacent only to a single component. The resulting graph is still a tree, but any leaf is a component of \( G \). Let \( K \) be any such leaf.

Let \( G_1 = K \) and let \( G_2 \) be the union of the remaining components of \( G \). Let \( G_1' \) and \( G_2' \) be the subgraphs of \( G' \) induced by \( f(E(G_1)) \) and \( f(E(G_2)) \), respectively. Then \( G_1' \) is connected by Condition (2). But \( G_2' \) is also connected, because \( G' \) is connected and \( G_1' \) has exactly one vertex in common with \( G_2' \). Let \( v' \) be the vertex of \( G'' \) common to \( G_1' \) and \( G_2' \).

We wish to apply the inductive assumption to both the functions \( f : E(G_i) \to E(G_i') \), for \( i = 1, 2 \). Thus, we must show that, for each of these, \( f \) is an algebraic duality and that Conditions (1), (2) and (3) hold. That \( f \) is an algebraic duality is proved exactly as in the proof of Theorem 2, so we omit this proof.

Condition (1) for both \( G_1' \) and \( G_2' \) is trivial, as is Condition (2) for \( G_1 \). Condition (2) for \( G_2 \) follows from Condition (2) for \( G \). Condition (3) for both is inherited from Condition (3) for \( G \) and the choice of the component \( K \).

Inductively, then, there are planar embeddings of \( G_1 \) and \( G_2 \) having \( G_1' \) and \( G_2' \), respectively, as planar duals and, in both cases, \( f \) is the geometric duality. Each of these embeddings may be chosen so that \( v' \) corresponds to the infinite face. Now placing \( G_1 \) and \( G_2 \) side by side yields the required embedding of \( G \).}

**References**


