THE COMPUTATIONAL COMPLEXITY OF DECIDING HAMILTONIAN-CONNECTEDNESS

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ABSTRACT. We determine the computational complexity of several decision problems related to hamiltonian-connected graphs. In particular, we show that it is NP-complete to determine whether a graph G is hamiltonian-connected. We also show that it is NP-complete to determine whether G is hamiltonian-connected from a distinguished vertex v. Lastly, we consider the complexity of multiple-solution and unique-solution variations.

SECTION 1. INTRODUCTION

The notion of a ***hamiltonian-connected graph***, in which there is a hamiltonian path between each pair of distinct vertices, was introduced in 1963 by Ore [11]. In 1981 Chartrand and Nordhaus [2] defined a variant of this idea: a graph G is ***hamiltonian-connected from a vertex*** v if there are hamiltonian paths from v to each other vertex of G. Hendry [5] studied graphs ***uniquely*** hamiltonian-connected from a vertex v, in which there is a ***unique*** hamiltonian path from v to each other vertex of G. While considerable work has been done on the structure of such graphs [2, 6, 10, 3], very little has been said about the computational complexity of the associated decision problems, despite the fact that the decision problems for hamiltonian and related graphs are among the most famous NP-complete problems.

In this paper we examine the complexity of several of these problems. We show in Section 2 that it is NP-complete to determine whether a graph is hamiltonian-connected. In Section 3 we show that it is NP-complete to determine if a graph is hamiltonian-connected from a vertex. In Section 4 we examine the complexity of multiply hamiltonian-connected problems, and, in Section 5, unique hamiltonian-connected problems.

SECTION 2. HAMILTONIAN-CONNECTED GRAPHS

In this section we will refer to the following decision problems.

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HCON	Instance: *A graph G.* Question: *Is G hamiltonian-connected?*
Huv	Instance: *A graph G and distinct vertices u, v.* Question: *Is there a hamiltonian path from u to v?*
Huv1	Instance: *A graph G and distinct, degree 1 vertices u, v.* Question: *Is there a hamiltonian path from u to v?*

These three problems are all obvious members of NP. The hamiltonian path problem, Huv, is NP-complete [4]. The NP-completeness of $Huv1$ follows easily, and is shown below. We then give a polynomial transformation from $Huv1$ to $HCON$, showing that $HCON$ is also NP-complete.

Lemma 1. *$Huv1$ is NP-complete.*

Proof. We give a transformation from Huv. Let (G, u, v) be an instance of Huv. Construct a new graph G' by appending degree 1 vertices, u' to u and v' to v. It is clear that (G, u, v) is a *yes*-instance of Huv if and only if (G', u', v') is a *yes*-instance of G'. □

Theorem 1. *$HCON$ is NP-complete.*

Proof. We give a transformation from $Huv1$. Let (G, u, v) be an instance of $Huv1$, so that u and v each have degree 1. Construct a new graph G_1 by adding two new vertices, v_1 and v_2, with v_1 adjacent to u, v, and v_2, and v_2 adjacent to all other vertices in G_1. We claim that G has a hamiltonian u-v path if and only if G_1 is hamiltonian-connected.

First, let H be a hamiltonian u-v path in G. For a vertex w in G, we will denote by w_u (resp. w_v) the neighbor of w on H that is closer to u (resp. v). We will also denote by $H(w, z)$ the section of H from the vertex w to the vertex z.

We show below how to use H to construct hamiltonian paths between each pair of vertices in G_1. We use x and y to denote vertices of G_1, not equal to u or v, with x closer than y to u on the hamiltonian path H.

(1)	x–y:	$H(x,u)$–v_1–$H(v,y_v)$–v_2–$H(x_v,y)$
(2)	x–u:	$H(x,v)$–v_1–v_2–$H(x_u,u)$
(3)	x–v:	Symmetric to Case 2
(4)	x–v_1:	$H(x,u)$–v_2–$H(x_v,v)$–v_1
(5)	x–v_2:	$H(x,v)$–v_1–$H(u,x_u)$–v_2
(6)	u–v:	u–v_1–v_2–$H(u_v,v)$
(7)	u–v_1:	$H(u,v)$–v_2–v_1
(8)	u–v_2:	$H(u,v)$–v_1–v_2
(9)	v–v_1:	Symmetric to Case 7
(10)	v–v_2:	Symmetric to Case 8
(11)	v_1–v_2:	v_1–$H(u,v)$–v_2

Thus, if G has a hamiltonian u-v path, then G_1 is hamiltonian-connected. Now assume that G_1 is hamiltonian-connected, and consider a v_1-v_2 hamiltonian path H_1 in G_1. Since v_1 is adjacent only to u, v, and v_2, we may assume without loss of generality that the edge v_1-u is on H_1. Since the edge v_1-v is not used, the vertex v becomes a degree 2 vertex, adjacent to v_2, so the edge v-v_2 must be on H_1 as well. Deleting the edges v_1-u and v-v_2 leaves a hamiltonian u-v path in G, completing the transformation from *Huvl* to *HCON*. □

SECTION 3. GRAPHS HAMILTONIAN-CONNECTED FROM A VERTEX

In this section we will refer to the following decision problems, each of which clearly belongs to *NP*. The hamiltonian cycle problem, *HAM*, was proven *NP*-complete by Karp in 1972 [9].

HCv Instance: *A graph G and distinguished vertex v.*
Question: *Is G hamiltonian-connected from v?*

HAM Instance: *A graph G.*
Question: *Does G have a hamiltonian cycle?*

Theorem 2. *HCv is NP-complete.*

Proof. We give a transformation from *HAM*. Let the graph G be an instance of *HAM*. Map G to a graph G' defined as follows:

(1) For each vertex x in G, G' contains a path of 4 vertices, $P_x = x_1$-x_2-x_3-x_4;
(2) For each edge x-y in G, G' contains the two edges x_1-y_4 and x_4-y_1;
(3) G' contains an additional vertex v, and, for each x in $V(G)$, v is adjacent to x_2 and x_3.

We claim that G is hamiltonian if and only if G' is hamiltonian-connected from v. First, suppose that G has a hamiltonian cycle C, and let x be a vertex in G. Note that C induces two natural hamiltonian cycles in $G' \setminus v$, one going through the corresponding 4-paths using edge x_4-y_1 in place of edge x-y, the other using edge x_1-y_4 in place of edge x-y. Notice also that each vertex x_i in $G' \setminus v$ is adjacent to a unique neighbor x_j of v, where x_j is on the same 4-path P_x as x_i (and $j = 2$ or 3). We can thus easily construct two different hamiltonian paths from v to x_i in G' by travelling from v to this neighbor x_j, and then around either one of the hamiltonian cycles induced by C in $G' \setminus v$, finishing at x_i.

Now suppose that G' is hamiltonian-connected from v. Consider any hamiltonian path H' starting at v. For some $x \in V(G)$, one of the two edges v-x_2, v-x_3 is on the path H'; assume without loss of generality that it is v-x_2. The next edge on H' is either x_2-x_1 or x_2-x_3. Noting that, for $y \neq x$, the entire 4-path P_y is on H', we can trace the path H' in each of these two cases, leading to the following possiblities:

(1) H' is a v-x_3 path, induced by a hamiltonian cycle in G;
(2) H' is a v-x_1 path, induced by a hamiltonian cycle in G;
(3) For some $y \neq x$, H' is a v-y_1 path, but H' is *not* induced by a hamiltonian cycle in G.

Since there must be a hamiltonian v-x_3 path for each $x \in V(G)$, we conclude that G has a hamiltonian cycle. □

Section 4. Multiple-solution Variations

For problems involving hamiltonian-connected graphs, there are two types of multiple-solution variations: one in which we require multiple hamiltonian paths for *some* pair of vertices, and one in which we require multiple hamiltonian paths for *all* pairs of vertices. In the list of problems that follows, we use the suffix "+" to indicate the former and the prefix "2" to indicate the latter. For example, a *yes*-instance of *HCON+* requires that there be a hamiltonian path between each pair of distinct vertices, and, for at least one pair of vertices, that there be two or more different hamiltonian paths; *2HCON* requires that there be two or more hamiltonian paths for *every* pair of distinct vertices. For the hamiltonian cycle problem *HAM*, we will use *2HAM* to denote the multiple-solution variation.

All of the problems given below are easily seen to be in NP. The NP-completeness of *2HAM* follows directly from a proof given in [8] of a result of Papadimitriou and Steiglitz [12] that it is NP-complete to determine if a graph G with a known hamiltonian cycle C has any other hamiltonian cycles.

HCON+	Instance: *A graph G.* Question: *Is G hamiltonian-connected in more than one way?*
2HCON	Instance: *A graph G.* Question: *For each pair of distinct vertices, x, y, are there two or more hamiltonian x-y paths?*
HCv+	Instance: *A graph G and distinguished vertex v.* Question: *Is G hamiltonian-connected from v in more than one way?*
2HCv	Instance: *A graph G and distinguished vertex v.* Question: *For each vertex $x \neq v$, are there two or more hamiltonian v-x paths?*
2HAM	Instance: *A graph G.* Question: *Does G have two or more hamiltonian cycles?*

Theorem 3. *The problems HCON+, 2HCON, HCv+, and 2HCv are all NP-complete.*

Proof. In the proof of Theorem 1, it is easy to construct, in each of the eleven cases, a different hamiltonian path, so that we get a transformation from *Huv1* to both *HCON+* and *2HCON*. Likewise, it is clear from the proof of Theorem 2 that the transformation from *HAM* may be viewed as a transformation either to *HCv+* or to *2HCv*. □

Section 5. Unique-solution Variations

In this section we will consider variants of earlier problems, adding the condition of *uniqueness*. In other words, a *yes*-instance requires that the given property be satisfied in exactly one way. For example, a *yes*-instance of *UHCv* requires, for each vertex $x \neq v$, that there is a unique hamiltonian v-x path.

UHCv	Instance: *A graph G and distinguished vertex v.*
	Question: *Is G uniquely hamiltonian-connected from v?*
UHAM	Instance: *A graph G.*
	Question: *Does G have a unique hamiltonian cycle?*

Note that we did not include a decision problem corresponding to uniquely hamiltonian-connected graphs: Hendry showed in [5] that such graphs can have at most three vertices.

In general, it is *not* clear that unique problems are members of NP, but they are natural members of D^p, which was defined in 1984 by Papadimitriou and Yannakakis [13] as the class of all decision problems that are the *difference* of two problems in NP. In other words, D^p consists of those problems that are the intersection of a problem in NP with a problem in *co-NP*. It is easy to see that $NP \cup coNP \subseteq D^p$.

Using the notation $\overline{\Pi}$ to denote the complement of a decision problem Π, we see that the above unique problems are in D^p:

$$UHCv = HCv \cap \overline{HCv+}$$
$$UHAM = HAM \cap \overline{2HAM}$$

Given a decision problem Π it is easy to show that $U\Pi$ is *co-NP* hard if one can find a polynomial transformation $\alpha : \Phi \to \Pi$, where Φ is NP-complete, such that for each *yes*-instance Y_Φ of Φ, every solution of $\alpha(Y_\Phi)$ — except for *one* — is induced by a solution of Y_Φ. If this is the case, then the transformation α takes $\overline{\Phi}$ to $U\Pi$. (See [7] for a discussion of results on unique problems.) For instance, this is done for *UHAM* in [8]. If the complement of an NP-complete problem is in NP, it's easy to show that $NP = co\text{-}NP$. Thus, the result of [12] gives the following.

Theorem 4 (Papadimitriou and Steiglitz [12]**).** *The problem UHAM is co-NP hard. If $UHAM \in NP$, then $NP = co\text{-}NP$.*

The transformation in the proof of Theorem 2 does not give the above result for *UHCv*. It is interesting to note that *yes*-instances of this problem (that is, graphs uniquely hamiltonian-connected from a vertex v) have many structural restrictions [5, 6, 10, 3]. For example, if the order n of G is greater than 3, then n is odd, $|E(G)| = (3n-3)/2$, and all vertices other than the distinguished vertex v have degree 2, 3, or 4. If graphs uniquely hamiltonian-connected from a vertex could be characterized in terms of a list of such structural restrictions, and if the list could be checked in polynomial time, then *UHCv* would be an element of P. On the other hand, very restricted forms of *HAM* continue to be NP-complete, e.g., *HAM* restricted to planar, bipartite graphs with degrees either 2 or 3 [8].

A problem Π is D^p-complete if any problem in D^p can be polynomially transformed to Π. If Π is D^p-complete, then it has the properties described in Theorem 4; in particular, if $\Pi \in NP$, then $NP = co\text{-}NP$. It is not known whether *UHAM* is D^p-complete. Indeed, Blass and Gurevich [1] have shown that it may be impossible to answer this question using standard techniques, by constructing two oracles, for which the relative D^p classes properly contain $NP \cup co\text{-}NP$, but such that the closely related problem $USAT$ (Given a boolean formula, does it have a unique satisfying truth assignment?) is D^p-complete relative to one oracle but not D^p-complete relative to the other.

We conclude with two open questions.

Open question 1. If $UHCv$ is in NP, then must $NP = co\text{-}NP$?

Open question 2. If the answer to the first question is *yes*, then does $UHCv$ have the stronger property of being D^p-complete?

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