MA 108, Review Sheet 2 Solutions

1. Let \( f(x) = 2x^3 + 1 \) and \( g(x) = \sqrt{3x + 7} \). Find \((f - g)(3)\) and \((f \cdot g)(-1)\).

\[
(f - g)(3) = f(3) - g(3) = (2(3^3) + 1) - \sqrt{9 + 7} = 55 - 4 = 51.
\]

\[
(f \cdot g)(-1) = f(-1)g(-1) = (2(-1)^3 + 1)(\sqrt{-3 + 7}) = (-1)(2) = -2.
\]

2. Let \( f \) and \( g \) be the functions in problem 1. Find formulas for \( f \circ g \) and \( g \circ f \), and find their domains.

\[
(f \circ g)(x) = f(g(x)) = f\left(\sqrt{3x + 7}\right) = 2\left(\sqrt{3x + 7}\right)^3 + 1.
\]

The domain of \( f \circ g \) is the set of all \( x \) such that \( g(x) \) is defined, and furthermore that \( f \) is defined at \( g(x) \). However, it is clear that \( f \) is defined for every real number input, so the domain for \( f \circ g \) is equal to the domain of \( g \) in this case. The domain of \( g \) is the set of \( x \) for which \( 3x + 7 \geq 0 \), which is the set \( x \geq -7/3 \), or \([-7/3, +\infty)\) in interval notation.

\[
(g \circ f)(x) = g(f(x)) = g(2x^3 + 1) = \sqrt{3(2x^3 + 1) + 7} = \sqrt{6x^3 + 10}.
\]

The domain of \( g \circ f \) is the set of all \( x \) such that \( f(x) \) is defined (which is all \( x \)) and such that \( g \) is defined at \( f(x) \). Recall that the domain of \( g \) is \([-7/3, +\infty)\), so for \( g \circ f \) to be defined at \( x \), we must have that \( f(x) \geq -7/3 \), or \( 2x^3 + 1 \geq -7/3 \), or \( 2x^3 \geq -10/3 \), or \( x^3 \geq -10/6 \), or \( x \geq \sqrt{-10/6} \).

3. Sketch (approximately) the graphs of the functions \( s = f + g \) and \( p = f \cdot g \), where \( f \) and \( g \) are the functions whose graphs are shown.

The graph of the sum function \( s = f + g \) is the parabola that passes through the point \((2,2)\). The graph of the product function \( p = f \cdot g \) is the cubic-shaped graph that crosses the x-axis at -2, 0, and 2.
4. Find the average rate of change of the function \( k(x) = x^2 - 5x + 3 \) on the intervals \([3, 5]\) and \([-3, 2]\). Illustrate the geometric meaning of these rates of change on the graph of the function.

The average rate of change of the function \( k \) on an interval \([a, b]\) is given by \( \frac{k(b) - k(a)}{b - a} \). So, the average rate of change on \([3, 5]\) is

\[
\frac{k(5) - k(3)}{5 - 3} = 3
\]

and the average rate of change on \([-3, 2]\) is

\[
\frac{k(2) - k(-3)}{2 - (-3)} = -6
\]

Geometrically, these average rates of change are the slopes of the secant lines connecting the points on the graph that lie above the interval endpoints in each case.

5. Find the instantaneous rate of change of the function \( k \) in the preceding problem at the point \( x = 4 \).

The instantaneous rate of change requested is given by

\[
\lim_{x \to 4} \frac{k(x) - k(4)}{x - 4} = \lim_{x \to 4} \frac{x^2 - 5x + 3 - (-1)}{x - 4} = \lim_{x \to 4} \frac{x^2 - 5x + 4}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+1)}{x-4} = \lim_{x \to 4} x + 1 = 5.
\]
6. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes. a) Find the average rate of change of the volume of water in the tank between \( t = 15 \) and \( t = 25 \) minutes. b) Estimate the instantaneous rate of change of the volume at \( t = 15 \) minutes.

Grid[{{t, 10, 15, 20, 25, 30}, {V, 440, 250, 111, 28, 0}}, Frame → All]

The average rate of change on [15, 25] is 
\[
\frac{V(25) - V(15)}{25 - 15} = \frac{28 - 250}{10} = \frac{-222}{10} = -22.2 \text{ gallons/min.}
\]

To estimate the instantaneous rate of change at \( t = 15 \), about the best we can do is to average the two average rates of change on [10, 15] and [15, 20], 
\[
\frac{V(15) - V(10)}{15 - 10} = \frac{250 - 440}{5} = \frac{-190}{5} = -38 \text{ gal/min.} \quad \frac{V(20) - V(15)}{20 - 15} = \frac{111 - 250}{5} = \frac{-139}{5} = -27.8 \text{ gal/min.}
\]

The average of these two values is 
\[
\frac{(-38) + (-27.8)}{2} = -32.9 \text{ gal/min.}
\]

7. Evaluate the following limits

- a) \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x+3)(x-1)}{x-1} = \lim_{x \to 1} (x + 3) = 4 \)

- b) \( \lim_{x \to 3} \frac{x^2 - 9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \left\{ \begin{array}{ll}
\lim_{x \to 3} (x + 3) = 6, & \text{if } x - 3 > 0 \\
\lim_{x \to 3} -(x + 3) = -6, & \text{if } x - 3 < 0
\end{array} \right. \)

So this second limit DNE, since the value of the limit depends on the direction of approach. If \( x > 3 \), then \( x - 3 > 0 \), so \( |x - 3| = x - 3 \). However, if \( x - 3 < 0 \), then \( |x - 3| = -(x - 3) \). Here is a plot:

Plot[f[x], {x, 0, 6}]
c) \[ \lim_{x \to \infty} \frac{3x^2 - 2x}{5x^2 + 1} = \lim_{x \to \infty} \frac{(3x^2 - 2x)(\frac{1}{x^2})}{(5x^2 + 1)(\frac{1}{x^2})} = \lim_{x \to \infty} \frac{3 - 2x}{5 + 1/x^2} = \frac{3}{5}. \]

\[ \text{Plot}\left[\left\{\frac{3x^2 - 2x}{5x^2 + 1}, \frac{3}{5}\right\}, (x, -10, 20), \text{PlotRange} \to (-1, 1)\right] \]

8. Find the derivative of the function \( f(x) = 3x^2 - 5x \) at the point \( x = a \). What is the instantaneous rate of change of the function at the point \( a = 3 \)?

The derivative of the function at \( x = a \), by definition, is given by the following limit:
\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h)^2 - 5(a+h) - (3a^2 - 5a)}{h} = \lim_{h \to 0} \frac{6ah + 3h^2 - 5h}{h} = \lim_{h \to 0} (6a + 3h - 5) = 6a - 5. \]

When \( a = 3 \), the instantaneous rate of change \( f'(3) = 6 \cdot 3 - 5 = 18 - 5 = 13 \).

9. Let \( f \) be the function whose graph is shown. Find the values of the following limits, and state where \( f \) is not continuous.
a) \( \lim_{x \to 0} f(x) = 2 \) (and \( \lim_{x \to 0} f(x) = 1 \)).

- b) \( \lim_{x \to 0} f(x) = \text{DNE} \). The function \( f \) is not continuous at \( x = 0 \).

- c) \( \lim_{x \to 1} f(x) = 2 \)

- d) \( \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \text{DNE} \) (there is no tangent line at \( x = 1 \)).

- e) \( \lim_{h \to 0} \frac{f(-2+h)-f(-2)}{h} = \text{tangent slope at } x = -2, \text{ which is } 0 \) (since the graph is horizontal).