Review Sheet 2 Solutions

1. If \( y = x^3 + 2x \) and \( \frac{dx}{dt} = 5 \), find \( \frac{dy}{dt} \) when \( x = 2 \).

We have that \( \frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 2 \frac{dx}{dt} = 3x^2 \cdot 5 + 2 \cdot 5 \), and this is equal to \( 3(2^2) \cdot 5 + 10 = 70 \) when \( x = 2 \).

2. A spherical balloon is being inflated so that its volume increases at a rate of 2 ft/min. How fast is the radius increasing when the diameter of the balloon is 4 ft across?

Let \( V(t) \) be the volume of the balloon at time \( t \). We are given that \( V'(t) = 2 \) in units of ft/min. Let \( r \) be the radius of the balloon (also a function of time). We want to know \( r' = \frac{dr}{dt} \) at the instant of time that the diameter (which is twice the radius) is 4 ft, or the radius is 2 ft. The formula for the volume of a sphere is \( V = \frac{4}{3} \pi r^3 \), so we have our relation between \( V \) and \( r \). Taking the derivative with respect to time, we have that \( V' = 4 \pi r^2 \frac{dr}{dt} \). Since \( V' = 2 \) (a constant), we can solve for \( \frac{dr}{dt} \) to get \( \frac{dr}{dt} = \frac{2}{4 \pi r^2} \); the units are ft\(^3\)/min. At the instant in time when \( r = 2 \), we obtain that \( \frac{dr}{dt} = \frac{2}{4 \pi \cdot 2^2} = \frac{1}{8 \pi} \) ft\(^3\)/min.

3. Find the critical points of the following functions: a) \( g(t) = |3t - 4| \) b) \( h(x) = \frac{x-1}{x^2+4} \).

Recall that the critical points (or critical numbers) of a function are the values \( x = a \) such that the derivative either equals 0 or fails to exist at \( a \). We know that the function \( g(t) = 3t - 4 \) whenever \( 3t - 4 \geq 0 \), and equals \(-3t + 4 \) whenever \( 3t - 4 < 0 \). In other words, the graph of \( g \) consists of two lines that meet in a V-shape at the point where \( 3t - 4 = 0 \), which is the point \( t = 4/3 \). That point is a critical point, because \( g' \) doesn’t exist at a "corner" point, and it is the only critical point, because everywhere else the slope of the curve is either 3 or -3. Here is a graph of \( g \):

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Plot[Abs[3 t - 4], {t, -1, 3}]
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The function $h$ is defined everywhere, because the denominator is always positive. Furthermore, the derivative exists everywhere, so the only critical points of $h$ are of the form $h'(x) = 0$. We can find the derivative using the quotient rule:

$$h'(x) = \frac{(x^2+4)(2x) - (2x+1)(2x)}{(x^2+4)^2}.$$ 

This fraction is 0 wherever its numerator is 0 (again, the denominator is never 0). So the critical points of $h$ are the solutions of $-x^2 + 2x + 4 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{-2} = 1 \pm \sqrt{5}$. Here is a graph of $h$ in which the two critical points are visible.

In[14]:= Plot[\(\frac{x-1}{x^2+4}\), {x, -3, 5}]

Out[14]=

4. Find global max and min values of $f(x) = x - \ln(x)$ on the interval [1/2, 2].

The extreme values will occur either at critical points or endpoints. The derivative is $f'(x) = 1 - 1/x$, which is defined everywhere on the interval. $f'(x) = 0$ at $x = 1$. Therefore, to find the extremes, we need to find the largest and smallest values among $f(1/2)$, $f(1)$, and $f(2)$.

In[15]:= \(f[x_] := x - \text{Log}[x]\);

In[17]:= N[f[1/2]]

Out[17]= 1.19315

In[18]:= N[f[1]]


In[19]:= N[f[2]]

Out[19]= 1.30685

We see the global max is 1.30685..., which occurs at $x = 2$, and the global min is 1, which occurs at $x = 1$. 
5. Repeat # 4 for \( f(x) = x \sqrt{4 - x^2} \) on \([-1, 2]\).

The function is defined and continuous on the given interval. The extreme values can occur only at critical points or endpoints. The derivative is \( f'(x) = \sqrt{4 - x^2} + x \frac{1}{2 \sqrt{4 - x^2}} (-2x) \). Note that \( f'(2) \) is not defined (due to division by 0), so \( x = 2 \) is a critical point (and also an endpoint). The derivative exists everywhere else on the given interval. Is the derivative ever equal to 0 on this interval?

\[
f'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = 0 \Rightarrow 4 - x^2 = x^2 \Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}.
\]

So there is one critical point in the interior of the interval, namely \( \sqrt{2} \). So, to find the max and min, we need to check the values \( f(-1) \), \( f(\sqrt{2}) \), and \( f(2) \).

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\text{In[22]:= } f[x_] := x \sqrt{4 - x^2}
\]
\[
\text{In[23]:= } f[-1]
\]
\[
\text{Out[23]= } -\sqrt{3}
\]
\[
\text{In[24]:= } f[\sqrt{2}]
\]
\[
\text{Out[24]= } 2
\]
\[
\text{In[25]:= } f[2]
\]
\[
\text{Out[25]= } 0
\]

Therefore, we see that the global max is 2, which occurs at \( x = \sqrt{2} \), and the global min is \(-\sqrt{3}\), which occurs at \( x = -1 \).
6. Find the point the MVT guarantees will exist for the function \( f(x) = \sqrt{x} \) on \([1, 9]\). Illustrate.

In[27]:= \( f[x_] := \sqrt{x} \)

The secant slope is \( \frac{\sqrt{9} - \sqrt{1}}{9 - 1} = \frac{2}{8} = \frac{1}{4} \). Since \( f \) is continuous on \([1, 9]\) and differentiable for all \( x > 0 \) (in particular, on \((1, 9)\), the MVT guarantees that there is at least one point \( c \) between 1 and 9 such that \( f'(c) = 1/4 \). \( f'(x) = \frac{1}{2 \sqrt{x}} \), and this equals \( 1/4 \) when \( x = 4 \). So the slope of the tangent line at \( x = 4 \) is equal to the slope of the secant line.

In[28]:= \( \text{Plot}[\{f[x], (1/4) (x - 4) + 2, (1/4) (x - 1) + 1\}, \{x, 0, 10\}] \)
7. The graph of $f''$ is shown. What are the x-coordinates of the points of inflection of $f$? On which intervals is $f$ concave up/down?

Points where $f''(x) = 0$ are candidates for points of inflection. To be a point of inflection, the second derivative has to change sign through the point, so the original function $f$ actually changes concavity at the point. Therefore, $x = 1$ and $x = 7$ are the points of inflection. On $(-\infty, 1)$, we have $f'' < 0 \Rightarrow f$ is concave down. On $(1, 7)$, we have $f'' > 0$ (except at $x = 4$), so $f$ is concave up on $(1, 7)$. Finally, on $(7, \infty)$, $f'' < 0 \Rightarrow f$ is concave down.

8. Let $f(x) = \frac{x^2}{x^2 + 3}$. Find (a) the interval(s) on which $f$ is increasing and decreasing; (b) the local maximum and minimum value(s) of $f$, and (c) the intervals of concavity and the point(s) of inflection.

The domain of $f$ is all real $x$, since the denominator is never 0. The derivative is $f'(x) = \frac{2x(x^2 + 3) - 2x^3}{(x^2 + 3)^2} = \frac{6x}{(x^2 + 3)^2}$, which exists everywhere and is 0 at $x = 0$. So there is one critical point, $x = 0$. Since $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, we have that $x = 0$ is a local minimum. Since everywhere else $f$ has positive outputs, we see that the global minimum of $f$ occurs at $x = 0$. There are no local maxima, since there is only one critical point. Furthermore, the sign of the first derivative to the left and right of 0 tells us that $f$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. It is clear that as $x$ tends to $\pm \infty$, $f(x) \to 1$, so the horizontal line $y = 1$ is a horizontal asymptote. Finally, we use the second derivative to locate intervals of concavity and points of inflection:

$$f''(x) = \frac{6(x^2 + 3)^2 - 2(x^2 + 3)(2x)(6x)}{(x^2 + 3)^4} = \frac{(x^2 + 3)(-18x^2 + 18)}{(x^2 + 3)^4}.$$ We see that $f''(x) = 0$ at $x = \pm 1$. Furthermore, on the interval $(-1, 1)$, we see $f'' > 0 \Rightarrow f$ is concave up, whereas on the intervals $(-\infty, -1)$ and $(1, \infty)$, $f'' < 0 \Rightarrow f$ concave down. Therefore, the points $(-1, 1/4)$ and $(1, 1/4)$ are points of inflection.

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In[47]:= f[x_] := x^2/(x^2 + 3)
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9. The graph of the first derivative $g'$ of a function $g$ is shown (on the review sheet). Identify the local extreme points and sketch a graph of $g$.

Local maxima occur where $g'(x) = 0$ and goes from positive to negative as you move to the right. Thus the local maxima are at $x = 1$ and $x = 5$. The point $x = 3$ is a local minimum, since the derivative is 0 at this point and $g'$ goes from negative to positive as we move to the right. A sketch will be omitted here.

10. Evaluate the following limits:

a) $\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)}$. Since the numerator and denominator each tend to 0, we can use L'Hospital's Rule: $\lim_{x \to 0} \frac{4\cos(4x)}{5\sec^2(5x)} = \frac{4}{5} = \frac{4}{5}.$

b) $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$. Since the numerator and denominator each tend to 0, we can try L'Hospital's Rule:

$\lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2}$. (We used L'Hospital's Rule twice.)

c) $L = \lim_{x \to \infty} x^{1/2}$. We first take logs: $\ln(L) = \lim_{x \to \infty} (1/x) \ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$. So $\ln(L) = 0 \Rightarrow L = e^0 = 1$.

11. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

Let the length and width be $l$ and $w$, respectively. Since $2l + 2w = 100$, we have $w = \frac{100-2l}{2}$. The area of the rectangle is $A = l \cdot w = l \cdot \left(\frac{100-2l}{2}\right) = 50l - l^2$. We need to maximize the area on the interval [0, 50]. (The smallest conceivable length is 0 and the largest is 50 m -- Why?) Taking the derivative, we find that $A'(l) = 50 - 2l$, which is 0 when $l = 25 \Rightarrow w = 25$. So the rectangle of largest area in this case is the 25m x 25m square.

12. A box with a square base and an open top must have a
volume of 32000 cm$^3$. Find the dimensions of such a box that has the smallest surface area.

Let $x$ = the length of the square base's side, and let $h$ = the height (both in cm). Then we must have a volume of 32000, so $x^2 h = 32000$, or $h = \frac{32000}{x^2}$. The surface area of the box includes the base and the four sides; therefore, the surface area $S = x^2 + 4 x h = x^2 + \frac{128000}{x}$; we must minimize this function on $(0,\infty)$ -- Why? The derivative is $S'(x) = 2 x - \frac{128000}{x^2}$; setting this equal to 0 and solving yields $x^3 = 6400 \Rightarrow x = 40$. So, the minimum surface area is attained with a box having a square base of side 40 cm and a height of 20 cm.