Final Review Answers

1. Evaluate each of the following limits:

- \[ \lim_{x \to 3} \frac{x^2 - 2x - 3}{3-x} = \lim_{x \to 3} \frac{(x+1)(x-3)}{(3-x)} = \lim_{x \to 3} - (x+1) = -4. \]
- \[ \lim_{t \to \infty} \frac{t^5 - 5 t^2 + 71.6}{6 t^{18.3} t + 21} = \lim_{t \to \infty} \frac{1/t^5 - 5/t^2 + 71.6/t^4}{6 - 18.3 t + 21/t^4} = 0/6 = 0. \] This could also be done by L’Hospital’s Rule.
- \[ \lim_{h \to 0} \frac{\sin(\pi/2) - h \sin(\pi/2)}{h} = \sin'(\pi/2) = \cos(\pi/2) = 0. \]

2. Find the derivatives of the following functions:

- \( f(x) = x^2 + 5x - 3 \). \( f'(x) = 2x + 5 \).
- \( g(x) = x^{(2/3)} \tan(x) \). \( g'(x) = (2/3) x^{(-1/3)} \tan(x) + x^{(2/3)} \sec^2(x) \).
- \( h(x) = \frac{(t+1)}{(t^2-1)} \). \( h'(x) = \frac{1(t^2-1)-2t(t+1)}{(t^2-1)^2} = \frac{-t^2-2t-1}{(t^2-1)^2} = \frac{-t(t+1)^2}{(t^2-1)^2} = \frac{-1}{(t-1)^2} \). An easier way to do this problem is to simplify first: \( h(x) = \frac{1}{t-1} = (t-1)^{-1} \), so \( h'(x) = -(t-1)^{-2} \).
- \( m(t) = (t^3 + \cos(t))^{12} \). \( m'(t) = 12(t^3 + \cos(t))^{11} \cdot (3t^2 - \sin(t)) \).
- \( n(x) = \arcsin(\tan(e^x)) \). \[ n'(x) = \frac{1}{\sqrt{1-(\tan(e^x))^2}} \cdot (\sec^2(e^x) \cdot (e^x))^1. \]

3. Find \( y' = \frac{dy}{dx} \) by implicit differentiation, where \( y \) is a function of \( x \) defined by the relation \( x y^3 + \sin(y) = 0 \).

We take the derivative on both sides of the relation with respect to \( x \), viewing \( y \) as a function of \( x \):

\[ y^3 + 3x y^2 y' + \cos(y) y' = 0. \]

Solving for \( y' \), we obtain

\[ y' = \frac{-y^3}{3x y^2 + \cos(y)}. \]
4. Evaluate the following using the FTC:

- \[ \int_1^3 \frac{x+7}{x} \, dx = \left[ \frac{1}{2}(x+7 \ln(x)) \right]_1^3 = (3 + 7 \ln(3)) - (1 + 7 \ln(1)) = 2 + 7 \ln(3). \]

- \[ \int_0^1 \cos(x) \, dx = \left[ \sin(x) \right]_0^1 = \sin(1) - \sin(0) = 0 - 0 = 0. \]

The graph of \( y = f(x) = \cos(x) \) on \([0, \pi]\) makes this result clear: one has two congruent regions whose signed areas "cancel out."

\[ y = \cos(x) \text{ on } [0, \pi] \]

- \[ \int_0^1 \frac{1}{1+x^2} \, dx = \arctan(x) \right|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \approx 0.707. \]

A graph shows this value is reasonable (note the rectangle has area 1).

\[ y = 1/(1+x^2) \]

5. Let \( f \) be the function whose graph is shown, and let \( F \) be defined by \( F(x) = \int_3^x f(t) \, dt \).

Answer the following questions:
a) Estimate $F(5)$ and $F(1)$:

$F(5) = \int_3^5 f(x) \, dx = \text{signed area of "triangle" on [3, 5]}$, which is $-\frac{1}{2} \cdot b \cdot h = -(1/2)(2)(4/3) = -4/3$.

$F(1) = \int_1^3 f(t) \, dt = -\int_3^5 f(t) \, dt = -4/3$, since the triangle on [1, 3] is congruent to the triangle on [3, 5].

b) Where is $F$ increasing? Explain your answer.

$F$ is increasing where $F' = f$ is positive, which is on the interval $(-3, 3)$. (Why is $F' = f$? By the first part of the Fundamental Theorem of Calculus!)

c) At which value(s) of $x$ does $F$ have local minima? Explain.

A local minimum of $F$ occurs at a critical point such that $F'$ is negative to the left and positive to the right. Since $F' = f$, the critical points are -3 and +3, the values of $x$ where $F'(x) = f(x) = 0$. Of these, -3 has $F'$ negative to the left and positive to the right, so $x = -3$ is a local minimum of $F$.

d) Where is $F$ concave down? Explain.

$F$ is concave down where $F' = f$ is decreasing, which is on the interval $(0, 5)$ -- or $(0, \infty)$, if trends continue as shown.

6. Referring to the function $f$ whose graph is given in the previous problem, sketch the graph of $g$, where $g(x) = f(x + 2) + 1$. Also sketch the graph of $F$ that was discussed in the previous problem.

The graph of $g$ is obtained by translating the graph of $f$ to the left by 2 units and vertically upward by 1 unit. We obtain:

Recall that $F$ is the antiderivative of $f$ that passes through the point $(3, 0)$. It has a local minimum at -3 and a local maximum at +3. Here is the graph:
7. Use the definition of the derivative (limit process) to compute \( h'(x) \), where \( h(x) = \frac{1}{x} \).

Then find the equation of the line tangent to the graph at the point \((2, \frac{1}{2})\).

By definition, \( h'(x) = \lim_{z \to x} \frac{h(z) - h(x)}{z - x} = \lim_{z \to x} \frac{\frac{1}{z} - \frac{1}{x}}{z - x} = \lim_{z \to x} \frac{x - z}{x(z - x)} = \lim_{z \to x} \frac{-1}{z} = \frac{-1}{x^2} \). So, the slope of the curve at \( x = 2 \) is \( h'(2) = -\frac{1}{4} \), and the equation of the tangent line is \( y - \frac{1}{2} = (-\frac{1}{4})(x - 2) \), or \( y = (-\frac{1}{4})x + 1 \).

Plot: \( \{x, \frac{1}{x}, (-\frac{1}{4})x + 1, (x, -2, 4)\} \)
8. Sketch a graph of a single function $f$ with the following properties:

- $f(0) = 1$, $f(1) = 4$, $f(-2) = 0$;
- $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$;
- $f'(-2) = f'(3) = 0$, and $f'(1)$ is not defined;
- $f'(x) > 0$ for $x < -2$, $-2 < x < 1$, and $x > 3$;
- $f'(x) < 0$ for $1 < x < 3$.

![Graph of a possible $y = f(x)$]

9. A cannonball is fired directly upward, starting from ground level (height = 0), at time $t = 0$ seconds. Let $h(t)$ represent the cannonball’s height above the ground level (in feet) at time $t$ seconds, $v(t) = 192 - 32 t$ the cannonball's vertical velocity (in feet per second) at time $t$, and $a(t)$ the vertical acceleration at time $t$ seconds.

a) Calculate $\int_{4}^{0} v(t) \, dt$. What does this answer tell you about the cannonball?

b) Showing all work, find a formula for $h(t)$.

c) Showing all work, find a formula for $a(t)$.

d) At what time is the cannonball at the highest point? How high is it at that time?

a) $\int_{4}^{0} v(t) \, dt = \int_{4}^{0} (192 - 32 t - 16 t^2) \, dt = \left[ 192 \cdot t - 16 \cdot t^2 \right]_{4}^{0} = \left[ 192 \cdot 4 - 16 \cdot 4^2 \right] - \left[ 192 - 16 \cdot 4^2 \right] = 512 - 176 = 336$. This is the displacement of the cannonball on the interval $[0, 4]$; that is, it is the net change in height of the cannonball on this interval. This is because $h(t)$ is an antiderivative of $v(t)$, so the integral equals $h(4) - h(1) = \text{final height} - \text{initial height}$.

b) As just observed, $h(t)$ is an antiderivative of $v(t)$, so $h(t) = 192t - 16t^2 + C$. Since $h(0) = 0$, we see $C = 0$; therefore, $h(t) = 192t - 16t^2$.

c) Since $a(t) = v'(t)$, we get that $a(t) = (192 - 32 t)' = -32$ (ft/sec)/sec.

d) The cannonball is at its highest point when it is instantaneously motionless: Solve $v(t) = 0$ to find the time of maximum height. $192 - 32t = 0 \Rightarrow t = 192/32 = 6$. So the maximum height is attained at time $t = 6$ sec, and $h(6) = 192 \cdot 6 - 16 \cdot 6^2 = 576$ ft is the maximum height attained.
10. Use the average of the left and right sums with four subintervals to approximate the value of $\int_{1}^{3} \frac{6}{x} \, dx$.

If we divide the interval $[1, 3]$ into four equal subintervals, their common length is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. The subinterval endpoints are $1 = 2/2, \ 3/2, \ 4/2, \ 5/2, \ \text{and} \ 6/2 = 3$. The left sum is therefore $(1/2) \cdot (f(1)+f(3/2)+f(2)+f(5/2))$, which is equal to 7.7:

```math
f[x_] := 6/x;
LeftSum[f, 1, 3, 4]
```

The sum = 7.7

The right sum is $(1/2) \cdot (f(3/2) + f(2) + f(5/2) + f(3))$, which is equal to 5.7:

```math
RightSum[f, 1, 3, 4]
```

The sum = 5.7

The average of the left and right sums is $(7.7+5.7)/2 = 6.7$. This should be pretty close to the exact area under the curve, and also pretty close to the Midpoint approximation:
\textbf{MidpointSum}[f, 1, 3, 4]

The sum = 6.53853

11. Evaluate the following using geometry:

- \( \int_{-2}^{1} |x| \, dx = -\int_{-2}^{1} x \, dx \)

From the graph, we see that \( \int_{-2}^{1} |x| \, dx \) = (area of left triangle) + (area of right triangle) = \((1/2)(2)(2) + (1/2)(1)(1) = 2.5\). Therefore, \( \int_{-2}^{1} |x| \, dx = -2.5 \).
\[ \int_{-1}^{2} (2 - 3x) \, dx \]

\[ \text{Plot}[2 - 3x, \{x, -1, 2\}, \text{PlotLabel} \rightarrow \text{"y = 2 - 3x"}] \]

The integral in this case is \((\text{area of left triangle}) - (\text{area of right triangle})\). The \(x\)-intercept of the line is \(x = 2/3\). So, the base of the left triangle is \(1 + 2/3\) and the base of the right triangle is \(2 - 2/3 = 4/3\). The height of the left triangle is \(2 - 3(-1) = 5\) and the height of the right triangle is \(-(2 - 3(2)) = -(-4) = 4\). So the integral is given by

\[
\frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1 + 2}{3} \right) \left( 5 \right) - \left( \frac{1}{2} \right) \left( \frac{4}{3} \right) \left( 4 \right)
\]

\[\frac{3}{2}\]

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12. Find the local maxima and minima of the function \(f\), given by \(f(x) = 3x^3 - 9x\), and the global maximum and minimum (outputs) of \(f\) on the interval \([-2, 3]\).

The domain of the function is all reals, and \(f\) is differentiable everywhere, so the only critical points are those of the form \(f'(x) = 0\). The derivative is \(f'(x) = 9x^2 - 9 = 9(x^2 - 1)\), which is 0 at \(x = \pm 1\). We see that the derivative is positive for \(x < -1\) and \(x > 1\), and negative on \((-1, 1)\), so \(f\) is increasing on \((-\infty, -1) \cup (1, \infty)\) and decreasing on \((-1, 1)\), so \(x = -1\) is a local max and \(x = 1\) is a local min. On the interval \([-2, 3]\), the global max and the global min can occur either at a critical point or an endpoint.

\[ f[x_] := 3x^3 - 9x \]

\[ \{f[-2], f[-1], f[1], f[3]\} \]

\[ \{-6, 6, -6, 54\} \]

We see that the global min output is -6, which occurs at \(x = -2\) and \(x = 1\), and the global max output is 54, which occurs at \(x = 3\). Here is the plot:
13. You wish to mail a cylindrical package whose combined length and girth (the circumference of a cross section perpendicular to the length) is 84 inches. What are the length and girth of the cylindrical tube with the largest volume that you can mail?

Let \( r \) be the radius of the package, and \( l \) be its length. The girth is then \( 2\pi r \), and we want as large a volume as possible, so we choose a package for which length + girth = \( l + 2\pi r = 84 \), so \( r = \frac{84-l}{2\pi} \). The volume of the package is (base area \( \times \) length), which is \( \pi r^2 l = \pi \left( \frac{84-l}{2\pi} \right)^2 l = \left( \frac{1}{4\pi} \right) (84^2 l - 168 l^2 + l^3) \). The domain is \([0, 84] \), clearly. Taking the derivative of the volume and setting it equal to 0, we obtain \( 84^2 - 336 l + 3 l^2 = 0 \), or \( 84 \cdot 28 - 112 l + l^2 = 0 \), or \( (l - 28) (l - 84) = 0 \), so the critical points are \( l = 28 \) and \( l = 84 \). From the factored form of the volume function, we see that the volume is 0 when \( l = 84 \) (and \( l = 0 \)), so the max volume occurs when \( l = 28 \) and (therefore) the girth = 56. So the maximum volume occurs when the girth is twice the length.

14. Compute the definite integral of the function \( h \), given by \( h(x) = \frac{1}{x^2} \), on the interval \([1/2, 5/2] \).

\[
\int_{1/2}^{5/2} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{1/2}^{5/2} = \left( -2/5 \right) - \left( -2/1 \right) = 2 - 2/5 = 8/5. \] (We used the Fundamental Theorem of Calculus.) We can check with Mathematica’s integration function:

\[
\int_{1/2}^{5/2} \frac{1}{x^2} \, dx
\]
15. Use linear approximation to estimate $\sqrt{9.01}$. [Hint: find the best linear approximation to $f(x) = \sqrt{x}$ at $x = 9$.]

The best linear approximation (or linearization) of $f$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. For us, $f(x) = \sqrt{x}$, so $f(9) = 3$ and $f'(x) = \frac{1}{2 \sqrt{x}}$, so $f'(9) = \frac{1}{2 \sqrt{9}} = \frac{1}{6}$. Therefore, $L(x) = 3 + \frac{1}{6}(x - 9)$, so $\sqrt{9.01} = f(9.01) \approx L(9.01) = 3 + \frac{1}{6}(9.01 - 9) = 3 + 0.01/6 = 3.001666...$. Here is the exact value computed by Mathematica to 10 decimal places:

$$\text{n}[-\sqrt{9 + 1/100}, 10]$$

$$3.001666204$$

Here is a picture of the function and its tangent line at $x = 9$:

$$\text{Plot}\left[\left\{\frac{\sqrt{x}, 3 + (1/6)(x - 9)}{16}\right\}, \{x, 1, 16\}, \text{PlotRange} \rightarrow \{-0.5, 4.5\}\right]$$

16. Find the point that the Mean Value Theorem guarantees will exist for the function $f(x) = \sqrt{x}$ on the interval $[1, 9]$.

The Mean Value Theorem requires that the function be continuous on the closed interval and differentiable on the open interval; this is clearly true for this function and interval. The point $c$ we seek satisfies $f'(c) = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$. So $f'(c) = \frac{1}{2 \sqrt{c}} = \frac{1}{4}$, so the solution is $c = 4$. The graph illustrates the geometric content: the slope of the chord equals the slope of the tangent at $(c, f(c))$. 
17. Sketch the graph of \( y = f(x) = \frac{x^3 - 12x + 5}{x^2 + 1} \) on the axes provided, given graphs of \( f' \) & \( f'' \). Note that can be used as a convenient "starting point."

The graph of \( f' \) tells you where \( f \) has critical points and is increasing/decreasing, and the graph of \( f'' \) tells you where \( f \) is concave up/down. The graph of \( f \) looks like this (I have added the line \( y = x \), which represents the "asymptotic" behavior of the function as \( x \to \pm \infty \)).