MA 111 Review Sheet 2 Answers

1. Find the derivative of each of the following functions

- $7x^5 + 4x^2 = 7x^5 + 4x^{-2}$. Ans: $35x^4 - 8x^{-3}$. Power rule used for each term.
- $\sin(\ln(3x))$. Ans: $\cos(\ln(3x)) \cdot \left(\frac{1}{3x}\right) \cdot 3 = \frac{\cos(\ln(3x))}{x}$. Chain rule used twice.
- $(3x^2 + 5)^4$. Ans: $4(3x^2 + 5)^3 \cdot (6x)$. Power rule and chain rule used.
- $\frac{\cos(x)}{x}$. Ans: $\frac{-\sin(x) \cdot 7^x - (7^x \ln(7) \cdot \cos(x))}{(7^x)^2}$. Quotient rule and trig. & exp. derivative facts.
- $\sqrt{3^2 + 5} = (3^2 x^5)^{(1/2)}$. Ans: $(1/2) \cdot (3^2 x^5)^{-1/2} \cdot (3^2 x^5) \cdot \ln(3) \cdot 2$. Power & chain rules used.
- $x^2 \arctan(2x)$. Ans: $(2x) \arctan(2x) + x^2 \cdot \left(\frac{1}{1 + (2x)^2}\right) \cdot 2$. Product & chain rules used.

2. Let $h(x) = e^{\sin(3x+1)}$. Find the equation of the tangent line to the graph of $h$ at $x = -1/3$.

The derivative is $h'(x) = e^{\sin(3x+1)} \cdot (\cos(3x + 1) \cdot 3)$. Plugging in $x = -1/3$, we find that $h'(-1/3) = e^{\sin(0)} \cdot (\cos(0) \cdot 3) = e^0 \cdot 1 \cdot 3 = 3$ is the slope of the tangent line at $x = -1/3$. The output at that point is $h(-1/3) = e^0 = 1$. So the equation of the tangent line is $y - 1 = 3(x - (-1/3))$, or $y = 3x + 2$.

3. Use implicit differentiation to find $y' = dy/dx$ if $y$ is defined implicitly by the relation $x^2 + y^2 = 2$. Then find the equation of the line tangent to the graph of the relation at the point $(1,1)$.

Taking the derivative with respect to $x$ on both sides, we find $2x + 2yy' = 0$. Solving for $y'$, we obtain $y' = \frac{-2x}{2y} = \frac{-x}{y}$. Plugging in $x = 1, y = 1$, we find that the slope of the curve at $(1,1)$ is $\frac{-2 \cdot 1}{1^2 + 2 \cdot 1} = -2/3$. Therefore, the equation of the tangent line at $(1,1)$ in point-slope form is $y - 1 = -(2/3)(x - 1)$. 
4. An object moves along the x-axis so that its position at time $t$ seconds is given by the function $p(t) = 4t^2 - 12t + 5$. (Suppose that the x-axis is scaled in meters.) What is the position of the object at time $t = 3$ seconds? What is its velocity and acceleration at that time? For what values of $t$ is the object’s position increasing?

$$p[t_] := 4 t^2 - 12 t + 5$$

The position of the object at $t = 3$ is:

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In[15]:= p[3]
Out[15]= 5
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The velocity function is the first derivative of the position function (units of meters/second), and the acceleration function is the second derivative of the position function (units of (meters/second)/second, or meters/second$^2$). Therefore,

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In[16]:= v[t_] := 8 t - 12
a[t_] := 8
In[18]:= v[3]
Out[18]= 12
In[19]:= a[3]
Out[19]= 8
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The object's position is increasing (moving to the right on the x-axis) when its velocity is positive. So we want to solve the inequality $v(t) > 0$. $8t - 12 > 0$ for $t > 12/8 = 3/2$. So, the object moves to the left until $t = 3/2$, at which point it is instantaneously motionless, and thereafter is moving to the right.

The graph of the position function makes this all transparent:

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In[20]:= Plot[p[t], {t, -2, 4}]
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Out[20]=
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5. For the amusement of the guests, some hotels have elevators on the outside of the building. One such hotel is 300 feet high. You are standing by a window 100 feet above the ground and 150 feet away from the hotel, and the elevator descends (from the top of the hotel) at a constant speed of 30 ft/sec, starting at time $t = 0$, where $t$ is time in seconds. Let $\theta$ be the angle between the horizontal and your line of sight to the elevator.

a) Draw a labeled diagram of the situation.

b) Find a formula for $h(t)$, the elevator’s height above the ground, as it descends from the top of the hotel.

c) Using your answer to part b), express $\theta$ as a function of time $t$ and find the rate of change of $\theta$ with respect to time. 

The rate of change of $\theta$ is a measure of how fast the elevator appears to you to be moving. At what height is the elevator when it appears to be moving fastest?

The height of the elevator above the ground is easily seen to be $h(t) = 300 - 30t$, since it is at height 300 at $t = 0$ and descends at 30 ft/sec.
One sees that \(\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{h(\theta) - 100}{150}\), so \(\theta = \arctan\left(\frac{h(\theta) - 100}{150}\right)\). The rate of change of \(\theta\) with respect to time is the derivative \(\theta' = \left(\frac{1}{1 + \left(\frac{h(\theta) - 100}{150}\right)^2}\right) \cdot \left(\frac{h'(\theta)}{150}\right) = \left(\frac{1}{1 + \left(\frac{100 - 100}{150}\right)^2}\right) \cdot (-30)\). (We used the formula for the derivative of \(\arctan\) and the chain rule; we then used the expression for \(h(t) = 300 - 30t\) to find its derivative is -30.)

The last question is to determine when \(\theta\) is changing most rapidly. This is the point when \(\theta'\) has maximum absolute value (note that \(\theta'\) is negative since \(\theta\) is decreasing). To make \(\theta'\) as large as possible in absolute value, we must make the denominator as small as possible, and this clearly happens when \(h(t) = 100\). So the angle \(\theta\) is changing most rapidly when the elevator reaches the 100-foot elevation, so that it is directly opposite the observer (this is intuitively clear -- think about it!).

6. Find the best linear approximation \(L(x)\) to the function \(f(x) = \sqrt{1 + x}\) near \(x = 0\). Is \(L(x)\) an overestimate or an underestimate of \(f(x)\)?

The linear approximation formula at \(s = a\) is \(L(x) = f(a) + f'(a) (x - a)\). In our case, \(a = 0\), \(f(0) = 1\), and \(f'(x) = \frac{1}{2 \sqrt{1 + x}}\), so \(f'(0) = \frac{1}{2}\). Therefore, the linear approximation \(L(x) = 1 + 1/2 \cdot (x - 0) = 1/2x + 1\). \(L(x)\) is an overestimate in this case, since the tangent line lies above the curve.

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\text{In[17]} = \text{Plot}\left[\{\sqrt{1 + x}, 1 + (1/2)x\}, \{x, -1, 2\}\right]
\]

\[
\text{Out[17]} = 
\]

7. Give an example of a function \(f\) with the property that \(f'\) is not equal to \(f\), but \(f''\) is equal to \(f'\).

\(f(x) = e^x + c\) for any nonzero constant works, since \(f'(x) = e^x \neq f(x)\), but \(f''(x) = (e^x)' = e^x = f'(x)\).
8. If a snowball melts so that its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 10 cm.

The surface area \( S = 4 \pi r^2 \), where \( r \) is the radius. Since the radius is half the diameter \( d \), we can rewrite the surface area as a function of the diameter: \( S(d) = \pi d^2 \). We WANT \( d' \) when \( d = 10 \); we are GIVEN \( S' = -1 \) in cm²/min. Differentiating the relation between \( S \) and \( d \), we obtain \( S' = 2 \pi d' \), so \( d' = \frac{S'}{2 \pi d} \). Plugging in \( d = 10 \) and \( S' = -1 \), we find that \( d' = \frac{-1}{2 \times 10} \approx -0.0159 \) cm/min. Therefore, the diameter is decreasing at a rate of .0159 cm/min when \( d = 10 \) cm.

9. Consider the functions \( f \) and \( g \) defined by the following graphs. Suppose that \( h(x) = g(f(x)) \) and \( j(x) = 3 f(x) + 2 \). Estimate the value of each of the following expressions:
   a) \( h(3) = g(f(3)) = g(3) = 0 \).
   b) \( h'(3) = g'(f(3)) \cdot f'(3) \approx g'(3) \cdot f'(3) \approx (-1) \cdot 6 = -6 \).
   c) \( j(-1) = 3 f(-1) + 2 = 3 \cdot 0 + 2 = 2 \).
   d) \( j'(-1) = 3 f'(-1) \approx 3 \cdot 4 = 12 \).
10. Sketch the graph of a function $g$ that has all of the following properties:
   - The domain of $g$ is $[-3, 5]$.
   - $g(3) = 0$; $g(0) = -2$; $g(5) = 2$; $g'(0) = 0$.
   - $g$ is continuous everywhere except at $x = 3$.
   - $\lim_{x \to 3^+} g(x) = 1$; $\lim_{x \to 3^-} g(x) = -1$.
   - $g'(x) < 0$ on $(-3, 0)$; $g'(x) > 0$ on $(0, 3)$ and $(3, 5)$. 

![Graph of function g](image)