Mathematics 113, Fall Term 2009
Topics List and Sample Problems for the Final Exam

This review sheet consists of two parts: a list of the topics covered in the course, and some sample problems. The final exam will cover the entire course, with perhaps a slight emphasis on later topics. **You may bring one double-sided 3 x 5 card of notes for use during the exam.**

**Exam 1 Topics** (text sections 5.4-5.5, 6.1-6.3, 6.5, and 7.1-7.2):

1. **Basic integration:** Antiderivatives, standard forms, the Fundamental Theorem of Calculus, finding antiderivatives by using algebra to rewrite the integrand as a sum of standard forms.
2. **Integration by substitution:** Rewriting an integral using a substitution of the form $u = g(x), du = g'(x)dx$ to obtain a simpler integral in terms of $u$. Updating $x$-limits to $u$-limits in a definite integral.
3. **Applications of integration:** areas of plane regions (usually defined by two or more graphs of functions); volumes (by slices or by cylindrical shells); average value of a function (including geometric interpretation); arc length.
4. **Integration by parts:** The integration by parts “formula.” Various integration by parts strategies, including lowering the power of $x$ and transposing a multiple of the desired integral.
5. **Trigonometric integrals:** Various trig identities (Pythagorean identities and double- and half-angle formulas) and their use in rewriting integrands involving trig functions.

**Exam 2 Topics** (text sections 7.3-7.4, 7.7-7.8, 9.1-9.3):

1. **Trigonometric substitutions:** Substitutions for handling expressions containing $\sqrt{x^2 - a^2}, \sqrt{x^2 + a^2},$ and $\sqrt{a^2 - x^2}$. Back substitutions.
2. **Integration of rational functions by partial fractions:** Rational functions, long division, recipes for partial fraction decomposition and integration of the resulting partial fractions.
3. **Approximate integration:** Approximate computation of definite integrals using the Trapezoidal, Midpoint, and Simpson approximations. Use of the error bounds to estimate errors and to predict the number of subintervals needed to achieve a desired accuracy. Use of methods with functions given as tables of data.
4. **Improper integrals:** The two cases of improper integrals: infinite intervals and infinite discontinuities. Convergence/divergence of improper integrals; computation of (convergent) improper integrals using limit process; comparison tests for convergence/divergence.
5. **Differential equations:** The concept of a DE. Modeling with DEs. Direction fields and Euler’s Method for finding approximate solutions. Separable DEs and solution via separation of variables and integration.

**Topics covered since second midterm** (10.1-10.2, 11.1-11.6, and 11.8-11.10 (portions)):

1. **Parametric curves:** Describing a curve by parametric equations. The calculus of parametric curves (slope of tangent line, instantaneous speed at a point, arc length and area calculations).
2. **Sequences and series:** Concept of a sequence. Convergence and divergence of sequences. Monotonic sequence theorem (a bounded increasing sequence converges). Infinite series, partial sums, and convergence/divergence of infinite series.
3. **Testing series for convergence:** the integral test (and estimation of remainders), comparison test, limit comparison test, ratio test, and root test for series with nonnegative terms.
4. **Absolute and conditional convergence.**
5. **Alternating series and the alternating series test.**
6. **Power series:** radius and interval of convergence, function represented by a power series, Taylor and Maclaurin series.

**Some sample problems:**

1. Evaluate the following:
   a. \[ \int \frac{\sin(\sqrt{x} + 1)}{\sqrt{x}} \, dx \]
   b. \[ \int x \cdot e^{2x} \, dx \]
   c. \[ \int \frac{x}{x^2 + 3x + 2} \, dx \]
   d. \[ \int_0^{\pi/2} \frac{\cos(t)}{\sqrt{3 \cdot \sin(t) + 1}} \, dt \]
   e. \[ \int \frac{1}{\sqrt{4 + x^2}} \, dx \]
   f. \[ \int \cos^3(x) \, dx \]
   g. \[ \int e^{-2x} \, dx \]
   h. \[ \sum_{n=2}^{\infty} \left( \frac{2}{5} \right)^n \]

2. Test the improper integral \[ \int_{3}^{\infty} \frac{x}{x^3 + 10} \, dx \] for convergence or divergence. If the integral converges, find a value \( d \) so that \[ \int_{d}^{\infty} \frac{x}{x^3 + 10} \, dx \] is less than .001.

3. Find the volumes generated when the region bounded by the curves \( y = e^x, \ x = 0, \ x = 1, \) and the \( x \)-axis is rotated about the \( x \)-axis and about the \( y \)-axis.

4. Let \( f(x) = \frac{3}{x} \). Write down, but do not evaluate, the integral that you would use to find the length of the graph of \( f \) over the interval \([1,3]\). Give a reasonable estimate of the length without performing the integral. Compare your estimate with the Right Riemann sum \( R_2 \) for the arc length integral (the subscript 2 means use two subintervals).
5. How many subintervals are needed to ensure that the Midpoint Sum $M_n$ is within .001 of the exact value of the integral $\int_0^2 (4 - x^2) \, dx$? [Recall that the error satisfies $|E_M| \leq \frac{K(b-a)^3}{24n^2}$.]

6. Find a formula for the $n$-th term of each of the following sequences:

   1/3, 2/5, 3/7, 4/9, 5/11, …  1, -1/2, 1/6, -1/24, 1/120, …

7. Give an example of:
   a. A sequence which converges monotonically to 2.
   b. A sequence which diverges to -infinity.
   c. A diverging p-series.
   d. A converging alternating geometric series.

8. Determine which of the following series converge. State clearly your reason for each answer.

   a. $\sum_{n=1}^{\infty} \frac{n+1}{n^{1/3}}$
   b. $\sum_{n=1}^{\infty} \frac{n}{5^n}$
   c. $\sum_{n=3}^{\infty} \frac{n^3}{n^2 + 1}$
   d. $\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{3k + 2}\right)$

9. For what values of $x$ does the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n}$ converge?

10. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point $P$ is twice the $y$-coordinate of $P$. Find an equation of the curve. What is the answer if the slope at $P$ is twice the $x$-coordinate of $P$?

11. A bacteria culture grows with a constant relative growth rate (i.e., $P' = kP$, where $P$ is the population at time $t$, and $k$ is a constant. After two hours there are 600 bacteria and after 8 hours the count is 75,000. a) Find the initial population (population at $t = 0$ hours). b) Find an expression for the population $P(t)$ after $t$ hours. c) Find the number of bacteria after 5 hours. d) Find the growth rate at $t = 5$ hours. e) When will the population reach 200,000 bacteria?

12. Find the points on the parametric curve $x = 10 - t^2$, $y = t^3 - 12t$ where the tangent is horizontal and vertical. Also find an equation of the tangent line at the point corresponding to $t = -1$.

13. Determine the radius and interval of convergence for each of the following power series:

   a. $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k \cdot 4^k}$
   b. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$
c. \( \sum_{i=1}^{\infty} \frac{(x + 5)^i}{i(i + 1)} \)

d. \( \sum_{m=1}^{\infty} \frac{2^m (x - 1)^m}{m} \)