Midterm Review 1 Solutions

1. Evaluate the following integrals

\[
\int_1^2 \left( \frac{y + 5}{y^3} \right) dy = \int_1^2 \frac{1}{y^2} dy + \int_1^2 \frac{5}{y} dy
\]

\[
\int_{-1}^{-1} (x - \sqrt{2 \cdot \text{Abs}[x]}) \, dx
\]

\[
y = x - 2|x|
\]

\[
\int_0^1 \left( x \left( \sqrt{x} + \sqrt{-x} \right) \right) \, dx = \int_0^1 x^{4/3} \, dx + \int_0^1 x^{5/4} \, dx
\]

2. Evaluate using an appropriate substitution

\[
\int \left( t^2 \cos \left( t^3 + 2 \right) \right) \, dt
\]

Let \( u = t^3 + 2, \, du = 3t^2 \, dt \). The integral becomes \( \frac{1}{3} \int \cos(u) \, du = \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin(x^3 + 2) + C \).
Let \( u = 3 + x, \) \( du = dx. \) \( u \) varies from \( 1 + 3 = 4 \) to \( 6 + 3 = 9. \) So the integral becomes \( \int_{u=4}^{9} u^{-1/2} \, du = 2 u^{1/2} \Big|_4^9 = 2 \cdot 3 - 2 \cdot 2 = 2. \)

Let \( u = e^x, \) \( du = e^x \, dx. \) Then the integral becomes \( \int \frac{1}{1 + e^x} \, du = \arctan(u) + C = \arctan(e^x) + C. \)

- **3. Find the area of the region...**

  \[
  \text{Plot}[[x^2, 2 x - 1], \{x, 0, 2\}]
  \]

  The region in question lies beneath \( y = x^2 \) and above the \( x \)-axis for \( 0 \leq x \leq 1/2, \) and then above the tangent line \( y = 2x - 1 \) for \( 0 \leq x \leq 1. \) The area is then

  \[
  \text{In}[108]=
  \int_0^{1/2} x^2 \, dx + \int_{1/2}^1 (x^2 - (2 x - 1)) \, dx
  \]

  \[
  \text{Out}[108]=
  \frac{1}{12}
  \]

- **4. Find the volume in two ways...**

  \[
  \text{Plot}[[\sqrt{x - 1}, \{x, -5, 5\}]
  \]
The region beneath the curve above the interval \([1, 5]\) is rotated about the \(y\)-axis. We want to find the volume in two ways.

**Disk method (or "washer" method).** The outer radius of the disks is 5, and the inner radius of the disks is obtained by solving \(y = \sqrt{x-1}\) for \(x: x = y^2 + 1\). The volume of the washer is \((\text{area}) \cdot (\text{thickness}) = \pi \left( 5^2 - (y^2 + 1)^2 \right) \, dy\), and the washers lie between \(y = 0\) and \(y = 2\). Therefore the volume is

\[
\int_0^2 \left( \pi \left( 25 - (y^2 + 1)^2 \right) \right) \, dy
\]

\[
\frac{544 \pi}{15}
\]

**Shell method.** The volume of the shells is given by \((\text{circumference}) \cdot (\text{height}) \cdot (\text{thickness}) = (2 \pi x) \left( \sqrt{x-1} \right) \, dx\), and we must add these up as the radius \(x\) ranges from 1 to 5. Therefore the volume is

\[
\int_1^5 \left( 2 \pi x \sqrt{x-1} \right) \, dx
\]

\[
\frac{544 \pi}{15}
\]

5. Describe a solid whose volume is equal to \(\int_0^1 (2 \pi (3 - y) (1 - y^2)) \, dy\).

The most likely interpretation of the integral is as a shell-method calculation, with the radius of the shell being \(3 - y\), the height being \(1 - y^2\), and the thickness being \(dy\). Since the thickness is \(dy\) this means that we are rotating our region about a horizontal axis (namely the horizontal line \(y = 3\)). The region we are rotating is bounded by the curve \(x = 1 - y^2\), or \(y = \sqrt{1-x}\), \(0 \leq x \leq 1\). The volume is then the ring obtained by rotating the region beneath the curve and above the \(x\)-axis, \(0 \leq x \leq 1\), about the line \(y = 3\).

```math
\text{Plot}\left[\left\{3, \sqrt{1-x}\right\}, \{x, 0, 1\}, \text{PlotRange \to \{-1, 6\}}\right]
```
6. Find the avg. value of \( f(t) = t \sin(t^2) \) on \([0, 10]\).

\[
\frac{1}{10 - 0} \int_0^{10} (t \sin(t^2)) \, dt
\]

0.00688406

The integral can be worked by hand using the substitution \( u = t^2 \).

7. The arc length of \( y = \sin(x^2) \) on \([0, \pi]\).

The arc length of the curve \( y = f(x) \) on \([a, b]\) is given by \( \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \). In our case, this formula yields an integral that Mathematica can't handle exactly.

\[
\int_{0}^{\pi} \sqrt{1 + (2x \cos(x^2))^2} \, dx
\]

\[
\int_{0}^{\pi} \sqrt{1 + 4x^2 \cos(x^2)^2} \, dx
\]

8. Some integration by parts examples

\[ \int_{0}^{1} \frac{y}{e^{y^2}} \, dy \]

\[
\int_{0}^{1} \frac{y}{e^{y^2}} \, dy
\]

\[
\frac{1}{4} - \frac{3}{4e^2}
\]

\[
N\left[ \frac{1}{4} - \frac{3}{4e^2} \right]
\]

0.148499

To do this integral by parts, write it as \( \int (ye^{-2y}) \, dy \) and "lower the power of \( y \)’: Let \( u = y, \, \, du = dy, \, \, dv = e^{-2y} \, dy, \, \, v = -\frac{1}{2} e^{-2y} \). Therefore,

\[
\int (ye^{-2y}) \, dy = y(-\frac{1}{2} e^{-2y}) - \int -\frac{1}{2} e^{-2y} \, dy, \, \, \text{etc...}
\]

\[ \int_{1}^{2} \left( x^4 \, (\log[x])^2 \right) \, dx \]

\[
\int_{1}^{2} \left( x^4 \, (\log[x])^2 \right) \, dx
\]

\[
\frac{2}{125} \left( 31 - 160 \log[2] + 400 \log[2]^2 \right)
\]
Here the best strategy is to let \( u = (\ln(x))^2 \), \( du = 2 \frac{\ln(x)}{x} \) \( dx \), \( dv = x^4 \) \( dx \), \( v = \frac{x^5}{5} \). The integration by parts formula then yields \( u \cdot v - \int v \cdot du = (\ln(x))^2 \cdot \frac{x^5}{5} - \int \frac{x^4}{5} \cdot (2 \frac{\ln(x)}{x}) \) \( dx \). Therefore, our answer (for the antiderivative) is equal to \( (\ln(x))^2 \cdot \frac{x^5}{5} - \frac{2}{5} \int x^4 \cdot \ln(x) \) \( dx \).

We can now tackle the remaining integral using integration by parts, with \( u = \ln(x) \), \( du = (dx/x) \), \( dv = x^4 \) \( dx \), \( v = \frac{x^5}{5} \). Therefore our final answer (before plugging in the limits) is
\[
(\ln(x))^2 \cdot \frac{x^5}{5} - \frac{2}{5} \left[ (\ln(x)) \cdot \frac{x^5}{5} - \frac{1}{5} \int x^4 \right] \cdot \ln(x) \) \( dx \) =
\[
(\ln(x))^2 \cdot \frac{x^5}{5} - \frac{2}{25} \ln(x) \cdot x^5 + \frac{2}{125} x^5.\]

Plugging in the limits and simplifying, one obtains the final numerical answer shown above.

\[ \int \sin(\sqrt{x}) \) \( dx \]
\[
\int \sin \left[\sqrt{x}\right] \) \( dx \]
\[
-2 \sqrt{x} \cos \left[\sqrt{x}\right] + 2 \sin \left[\sqrt{x}\right]
\]

Here the trick is to let \( u = \sqrt{x} \) (or \( x = u^2 \), \( du = \frac{dx}{2 \sqrt{x}} \), or \( dx = 2u \) \( du \). Therefore, the integral becomes \( 2 \int \sin(u) \cdot u \) \( du \). At this point, an easy integration by parts, with \( w = u \), \( dv = \sin(u) \) \( du \), \( dw = du \), \( v = -\cos(u) \), yields the answer shown above, after replacing \( u \) by \( \sqrt{x} \).

\section{9. Evaluate} \( \int_{0}^{\pi/3} \tan^5(x) \sec^5(x) \) \( dx \)

The trick here is to split off a \( \sec^2(x) \) and rewrite the rest in terms of the tangent. We get \( \int_{0}^{\pi/3} \tan^5(x) \sec^4(x) \) \( (\sec^2(x) \) \( dx \) =
\[
\int_{0}^{\pi/3} \tan^5(x) \left(\tan^2(x) + 1\right) \) \( \sec^2(x) \) \( dx \).\] Now we let \( u = \tan(x) \), \( du = \sec^2(x) \) \( dx \), and the \( u \)-limits run from 0 to \( \sqrt{3} \). Therefore the answer is given by the following "elementary" integral.
\[
\int_{0}^{\sqrt{3}} \left( u^5 \left( u^2 + 1 \right) \right) \) \( du \]
\[
\frac{981}{20} \]
\[
N[\frac{981}{20}] \]
\[
49.05 \]