MA 113, Fall 2009
Midterm 2 Review Solutions

1. Evaluate the following integrals (trig substitution examples)

- \( \int_0^\sqrt{3} \frac{x^2}{\sqrt{16-x^2}} \, dx \)

Here we use \( x = 4 \sin(\theta) \), \( dx = 4 \cos(\theta) \, d\theta \). Then the integral (ignoring limits for the moment) becomes
\[
\int \frac{64 \sin^3(\theta)}{4 \cos(\theta)} \cdot 4 \cos(\theta) \, d\theta = 64 \int \sin^3(\theta) \cdot \sin(\theta) \, d\theta.
\]

Letting \( u = \cos(\theta) \), the integral becomes
\[
-64 \int (1-u^2) \, du = -64 \left( u - \frac{u^3}{3} \right).
\]

Now let’s update the limits: When \( x = 0 \), \( \theta = \arcsin(x/4) = \arcsin(0) = 0 \), and when \( \theta = 0 \), \( u = \cos(0) = 1 \). When \( x = 2 \sqrt{3} \), \( \theta = \arcsin \left( \frac{\sqrt{3}}{4} \right) = \arcsin \left( \frac{\sqrt{3}}{2} \right) = \pi/3 \), and when \( \theta = \pi/3 \), \( u = \cos(\pi/3) = 1/2 \). Therefore, the definite integral has the value
\[
-64 \left( 1 - \frac{1}{2^3} \right) \left|^{\sqrt{3}}_{0} \right. = 40/3.
\]

- \( \int \frac{x}{\sqrt{x^2-7}} \, dx \)

If we use a trig substitution, we let \( x = \sqrt{7} \sec(\theta) \), \( dx = \sqrt{7} \sec(\theta) \tan(\theta) \, d\theta \).

The integral becomes
\[
\int \frac{\sqrt{7} \sec(\theta)}{\sqrt{7} \sec(\theta)} \tan(\theta) \, d\theta = \sqrt{7} \int \sec^2(\theta) \, d\theta = \sqrt{7} \tan(\theta) = \sqrt{7} \cdot \sqrt{x^2-7} = \sqrt{x^2-7} + C.
\]

A more direct approach in this case is to let \( u = x^2 - 7 \), \( du = 2x \, dx \), so the integral becomes \( \frac{1}{2} \int u^{-1/2} \, du \).

- \( \int_0^3 \sqrt{x^2+4} \, dx \)

If a trig substitution is used, it would be \( x = 2 \tan(\theta) \), \( dx = 2 \sec^2(\theta) \, d\theta \). The limits on \( \theta \) would then be from 0 to \( \arctan(\frac{1}{2}) \). The integral becomes
\[
\int_0^{\arctan(1/2)} 2 \tan(\theta) \cdot 2 \sec^2(\theta) \cdot 2 \sec^2(\theta) \, d\theta = 8 \int_0^{\arctan(1/2)} \sec^3(\theta) \cdot \sec(\theta) \tan(\theta) \, d\theta =
\]

\[8\int_{u=1}^{\sqrt{5}\,/\,2} u^2 \, du = 8 \left( \frac{u^3}{3} \right) \bigg|_{1}^{\sqrt{5}\,/\,2} = \frac{5^{3/2}}{3} - \frac{8}{3}.
\]

A more direct approach is to let \( u = x^2 + 4 \), \( du = 2x \, dx \), so that the integral becomes \( \frac{1}{2} \int_4^{\sqrt{5}\,/\,2} u^{1/2} \, du = \frac{1}{3} (5^{3/2} - 4^{3/2}) = \frac{5^{3/2}}{3} - \frac{8}{3} \).

2. Integrate \( \int \frac{x^2}{(x-3)(x+2)^2} \, dx \)

The partial fractions setup is
\[
\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.
\]

One then determines the values of \( A \), \( B \), and \( C \):
\[
\text{Apart}\left[\frac{x^2}{(x-3)(x+2)^2}\right]
\]

\[
\frac{9}{25 \left(-3 + x\right)} - \frac{4}{5 \left(2 + x\right)^2} + \frac{16}{25 \left(2 + x\right)}
\]

The answer is \( A = \frac{9}{25}, \ B = \frac{16}{25}, \ \text{and} \ C = -\frac{4}{5}. \)

The integral then becomes
\[
\int \frac{9}{25 \left(x-3\right)} \, dx + \int \frac{16}{25 \left(x+2\right)} \, dx - \int \frac{4}{5 \left(x+2\right)^2} \, dx,
\]
which yields:
\[
(9/25) \ln \left| x-3 \right| + (16/25) \ln \left|x+2\right| + (4/5)/(x+2) + C.
\]

3. \( L = \int_{4}^{5} \sqrt{\frac{x}{2}} \, dx \)

The midpoint rule will give an overestimate of this integral. (The tangent lines at the midpoint lie above the curve, which is concave down.)

How many subintervals are needed to ensure that the midpoint method yields an error \( \leq .00001 \)? We know that this error is
\[
|E_M| = |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2},
\]
where \( K_2 \) is an upper bound for the absolute value of the second derivative of the integrand on the interval of integration. Our second derivative is \(-1/4 \cdot x^{-3/2}\). It is clear from the graph that we can use .25 as our upper bound.

\[
\text{Plot}\left[\left(1/4\right) \cdot x^{(-3/2)}, \{x, 1, 4\}, \text{PlotRange} \to \{-1, .3\}\right]
\]

Therefore, the error incurred when using the midpoint method with \( n \) subintervals is bounded by \( \left(\frac{25 \cdot (4-1)^3}{24 \cdot n^2}\right) = \frac{27}{96 \cdot n^2} \). We need to solve the inequality \( \frac{27}{96 \cdot n^2} \leq .00001 \Rightarrow \frac{96 \cdot n^2}{27} \geq 10^5 \).

\[
\text{Solve}\left[\frac{96 \cdot n^2}{27} = 10^5, \ n\right]
\]

\[
\left\{\{n \rightarrow -75 \sqrt{5}\}, \{n \rightarrow 75 \sqrt{5}\}\right\}
\]

\[
\text{N}\left[75 \sqrt{5}\right]
\]

167.705
We see that 168 subintervals will guarantee the desired accuracy.

The trapezoidal rule will be an underestimate in this case, and the number of subintervals needed to ensure the desired accuracy is found using the same kind of error bound, where the constant 24 is replaced by 12. Therefore, one obtains the inequality 

\[
\frac{48n^2}{27} \geq 10^5.
\]

Solve \[ \frac{48n^2}{27} = 10^5, \] \n\{ \{n \rightarrow -75 \sqrt{10} \}, \{n \rightarrow 75 \sqrt{10} \} \}

\[ n \rightarrow 75 \sqrt{10} \]

237.171

Thus we would need to use 238 subintervals to ensure the desired accuracy with the trapezoidal rule. Let's see what the values actually are:

\[ f[x_] := \sqrt{x} \]
\[ \frac{14}{3} \]

\[ N[RiemMid[f, 1, 4, 168]] \]
4.66667

\[ N[Abs[14/3 - RiemMid[f, 1, 4, 168]]] \]
\[ 3.3216 \times 10^{-6} \]

\[ N[RiemTrap[f, 1, 4, 238]] \]
4.66666

\[ N[Abs[14/3 - RiemTrap[f, 1, 4, 168]]] \]
\[ 6.64323 \times 10^{-6} \]

We see that both errors are within the desired upper bound of \(10^{-5}\).

4. Use Simpson's Rule on the given data.

Since the total change in the velocity (from 0 to the final velocity) is equal to the integral of the acceleration, we see that the velocity at \(t = 6\) is equal to the integral of the acceleration on the interval [0,6]. In this case, we have seven equally-spaced data points, which gives us six subintervals (an even number) of width \(\Delta x = 1\). Therefore, the Simpson approximation to the integral is given by

\[ \frac{1}{3} (1 * 0 + 4 * 1 + 2 * 4 + 4 * 10 + 2 * 13 + 4 * 9 + 1 * 0) \]

38.
5. Determine if the improper integrals converge or diverge. If convergent, find the value of the integral.

a) \( \int_{-1}^{0} \frac{1}{x^2} \, dx \) This is improper at the upper endpoint.

By definition, this is \( \lim_{b \to 0^-} \left( \int_{-1}^{b} \frac{1}{x^2} \, dx \right) = \lim_{b \to 0^-} \left( \left[ -\frac{1}{x} \right]_{-1}^{b} \right) = \lim_{b \to 0^-} \left( -\frac{1}{b} + 1 \right) = +\infty \). Diverges.

b) \( \int_{-3}^{3} \frac{2}{x^2+1} \, dx \) This integral is not improper!

c) \( \int_{0}^{2} \frac{x-3}{2x-3} \, dx \) The integrand blows up at \( x = 3/2 \), so this integral is improper. There are two directions of approach to the "bad" point 3/2 on the interval [0, 2].

\[ \lim_{x \to 3/2^-} \int_{0}^{2} \frac{x-3}{2x-3} \, dx. \] The integrand is an improper rational function, so we begin by dividing out: \( \frac{x-3}{2x-3} = \frac{1}{2} - \frac{3/2}{2x-3} \). So we investigate \( \lim_{x \to 3/2^-} \left( \frac{1}{2} x - \frac{3}{4} \ln(2x-3) \right) \). This limit will blow up, since \( \ln(2x-3) \) tends to \(-\infty\) as \( c \to 3/2 \). Thus this improper integral diverges.

d) \( \int_{0}^{\infty} \frac{x}{(x^2+2)^2} \, dx \) Note that the integrand is less than \( \frac{x}{x^4} = \frac{1}{x^3} \) for positive \( x \). We know that \( \int_{1}^{\infty} \frac{1}{x^3} \, dx \) is convergent, hence the given integral is convergent.

To compute the integral: \( \lim_{b \to +\infty} \int_{0}^{b} \frac{1}{x^2+2} \, dx = \lim_{b \to +\infty} \frac{1}{2} \int_{u=2}^{b^2+2} \frac{1}{u^2} \, du = \lim_{b \to +\infty} \left( \frac{1}{4} - \frac{1}{2(b^2+2)} \right) = \frac{1}{4} \). Mathematica agrees:

\[ \int_0^{\infty} \frac{x}{(x^2+2)^2} \, dx \]
\[ \frac{1}{4} \]

6. Estimate \( y(.4) \), where \( y' = 2x y^2 \), \( y(0) = 1 \), using Euler's Method with \( \Delta x = .2 \) and \( .1 \).

Compare with the exact value.

\[ f[x_{\text{old}}, y_{\text{old}}] := 2 \, x \, y^2 \]
oldX = 0; oldY = 1;
\Delta x = .1;
slope = f[oldX, oldY];
newX = oldX + \Delta x;
newY = oldY + slope * \Delta x;
Print["slope = ", slope, " new X = ", newX, " new Y = ", newY];
oldX = newX; oldY = newY;

Here is the table of values when \( \Delta x = .2 \) -- we see that \( y(.4) \approx 1.08 \).
Here is the table of values when $\Delta x = .1$ -- we see $y(.4) = 1.12924$ is a better approximation.

\[
\begin{pmatrix}
\text{oldX} & \text{oldY} & \text{slope} & \text{newX} & \text{newY} \\
0 & 1 & 0 & .2 & 1 \\
.2 & 1 & .4 & 1 & 1.08
\end{pmatrix}
\]

To find the exact solution, we separate the variables and integrate:

\[
\int \frac{1}{y^2 + 1} \, dy = \int 2 \, dx \Rightarrow \int \frac{-1}{y} = x^2 + C \Rightarrow y = \frac{1}{\sqrt{C-x^2}}.
\]

To satisfy the initial condition $y(0) = 1$, we must have $C = 1$, so the exact solution is $y(x) = \frac{1}{\sqrt{1-x^2}}$. The exact output when $x = .4$ is:

\[
\frac{1}{1 - .16} = 1.19048
\]

**7. Find the solution of the DE $y' = y^2 + 1$ that satisfies $y(1) = 0$.**

\[
\int \frac{1}{y^2 + 1} \, dy = \int 2 \, dx \Rightarrow \arctan(y) = x + C \Rightarrow y = \tan(x + C).
\]

If $y(1) = 0$ is to hold, we must have $0 = \tan(1 + C) \Rightarrow C = -1$. So the exact solution is $y = \tan(x - 1)$. Here is a picture of the direction field and the solution curve passing through $(1,0)$. 

\[\text{Picture of direction field and solution curve}\]