1. \( I = \int_{1}^{4} \sqrt{x} \, dx \)

The midpoint rule will give an overestimate of this integral. (The tangent lines at the midpoint lie above the curve, which is concave down.)

How many subintervals are needed to ensure that the midpoint method yields an error \( \leq .00001 \)? We know that this error is

\[
|E_M| = |I - M_n| \leq \frac{K_2(b-a)^3}{24 n^2},
\]

where \( K_2 \) is an upper bound for the absolute value of the second derivative of the integrand on the interval of integration. Our second derivative is \(-\frac{1}{4}x^{-3/2}\). It is clear from the graph that we can use \(.25\) as our upper bound.

\[
\text{Plot}[(1/4) x^{-(3/2)}, \{x, 1, 4\}, \text{PlotRange} \to \{-1, 3\}]
\]

Therefore, the error incurred when using the midpoint method with \( n \) subintervals is

\[
\frac{27}{96 n^2} \leq .00001 \Rightarrow \frac{96 n^2}{27} \geq 10^n.
\]

\[
\text{Solve}\left[\frac{96 n^2}{27} = 10^6, \quad n\right]
\]

\[
\{\{n \to -75 \sqrt{5}\}, \{n \to 75 \sqrt{5}\}\}
\]

\[
\lfloor 75 \sqrt{5} \rfloor = 167.705
\]

We see that 168 subintervals are needed to achieve the desired accuracy.

The trapezoidal rule will be an underestimate in this case, and the number of subintervals needed to ensure the desired accuracy is found using the same kind of error bound, where the constant 24 is replaced by 12. Therefore, one obtains the inequality
The trapezoidal rule will be an underestimate in this case, and the number of subintervals needed to ensure the desired accuracy is found using the same kind of error bound, where the constant $24$ is replaced by $12$. Therefore, one obtains the inequality

$$\frac{48 n^2}{27} \geq 10^5.$$ 

Solve $\frac{48 n^2}{27} = 10^5$, $n$

$$\left\{ \begin{array}{l} n \to -75 \sqrt{10} \\ n \to 75 \sqrt{10} \end{array} \right\}$$

$N \left[ 75 \sqrt{10} \right]$

$237.171$

Thus we would need to use 238 subintervals to ensure the desired accuracy with the trapezoidal rule. Let's see what the values actually are:

$$\frac{f[x]}{x} := \sqrt{x}$$

$$\int_{1}^{4} f[x] \, dx$$

$$\frac{14}{3}$$

$N[RiemMid[f, 1, 4, 168]]$

$4.66667$

$N[Abs[14/3 - RiemMid[f, 1, 4, 168]]]$

$3.3216 \times 10^{-6}$

$N[RiemTrap[f, 1, 4, 238]]$

$4.66666$

$N[Abs[14/3 - RiemTrap[f, 1, 4, 168]]]$

$6.64323 \times 10^{-6}$

We see that both errors are within the desired upper bound of $10^{-5}$.

### 2. Use Simpson's Rule on the given data.

In this case, we have seven equally-spaced data points, which gives us six subintervals (an even number) of width $\Delta x = 1$. Therefore, the Simpson approximation is given by

$$N \left[ \frac{1}{3} (1 \cdot 0 + 4 \cdot 1 + 2 \cdot 4 + 4 \cdot 10 + 2 \cdot 13 + 4 \cdot 9 + 1 \cdot 0) \right]$$

$38$. 
3. Determine if the improper integrals converge or diverge. If convergent, find the value of the integral.

a) \( \int_{-1}^{4} \frac{1}{x^3} \, dx \) This is improper at the upper endpoint.

By definition, this is \( \lim_{b \to 0^-} \left( \int_{-1}^{b} \frac{1}{x^3} \, dx \right) + \lim_{b \to 0^+} \left( \int_{b}^{4} \frac{1}{x^3} \, dx \right) \). Hence, the integral diverges.

b) \( \int_{-3}^{3} \frac{2}{x^2 + 1} \, dx \) This integral is not improper!

c) \( \int_{0}^{3} \frac{x - 3}{2x^3 - 3} \, dx \) The integrand blows up at \( x = 3/2 \), so this integral is improper. There are two directions of approach to the "bad" point 3/2 on the interval [0, 2].

\[ \lim_{x \to \infty} \left( \int_{0}^{3/2} \frac{x - 3}{2x^3 - 3} \, dx \right) \]

The integrand is an improper rational function, so we begin by dividing out: \( \frac{x - 3}{2x^3 - 3} = \frac{1}{2} - \frac{3}{2x^3 - 3} \). So we investigate \( \lim_{x \to \infty} \left( \frac{1}{2} x - \frac{3}{2} \ln(2x^3 - 3) \right) \). This limit will blow up, since \( \ln(2 - c) \) tends to \( -\infty \) as \( c \to 3/2 \). Thus this improper integral diverges.

d) \( \int_{0}^{\infty} \frac{x}{(x^2 + 2)^2} \, dx \) Note that the integrand is less than \( \frac{1}{x^3} \) for positive \( x \). We know that \( \int_{0}^{\infty} \frac{1}{x^3} \, dx \) is convergent, hence the given integral is convergent.

To compute the integral: \( \lim_{b \to \infty} \int_{0}^{b} \frac{x}{(x^2 + 2)^2} \, dx = \lim_{b \to \infty} \frac{1}{2} \int_{0}^{b} \frac{1}{u^2} \, du = \lim_{b \to \infty} \left( \frac{1}{2} \ln(b^2 + 2) \right) = \frac{1}{4}. \) Mathematica agrees:

\[ \int_{0}^{\infty} \frac{x}{(x^2 + 2)^2} \, dx = \frac{1}{4} \]

4. Estimate \( y(\cdot.4) \), where \( y' = 2x y^2 \), \( y(0) = 1 \), using Euler's Method with \( \Delta x = .2 \) and .1. Compare with the exact value.

\[ \text{In}[46]:= f[x_, y_] := 2 \times y^2 \]
\[ \text{In}[47]:= \text{oldX} = 0; \text{oldY} = 1; \]
\[ \text{In}[48]:= \Delta x = .1; \]
\[ \text{In}[64]:= \text{slope} = f[\text{oldX}, \text{oldY}]; \]
\[ \text{newX} = \text{oldX} + \Delta x; \]
\[ \text{newY} = \text{oldY} + \text{slope} \times \Delta x; \]
\[ \text{Print}[[\text{slope} = \text{oldX}, \text{oldY}]; \]
\[ \text{oldX} = \text{newX}; \text{oldY} = \text{newY}; \]

Here is the table of values when $\Delta x = .2$ -- we see that $y(.4) \approx 1.08$.

\[
\begin{array}{cccccc}
\text{oldX} & \text{oldY} & \text{slope} & \text{newX} & \text{newY} \\
0 & 1 & 0 & .2 & 1 \\
.2 & 1 & .4 & .4 & 1.08
\end{array}
\]

Here is the table of values when $\Delta x = .1$ -- we see $y(.4) = 1.12924$ is a better approximation.

\[
\begin{array}{cccccc}
\text{oldX} & \text{oldY} & \text{slope} & \text{newX} & \text{newY} \\
0 & 1 & 0 & .1 & 1 \\
.1 & 1 & .2 & .2 & 1.02 \\
.2 & 1.02 & .41616 & .3 & 1.0616 \\
.3 & 1.0616 & .676217 & .4 & 1.12924
\end{array}
\]

To find the exact solution, we separate the variables and integrate:

\[
\int \frac{1}{y^2} \, dy = \int 2 \, dx \Rightarrow \frac{-1}{y} = x^2 + C \Rightarrow y = \frac{1}{C-x^2}.
\]

To satisfy the initial condition $y(0) = 1$, we must have $C = 1$, so the exact solution is $y(x) = \frac{1}{1-x^2}$. The exact output when $x = .4$ is:

\[
\ln(1) = \frac{1}{1-.16} = 1.19048
\]

5. Find the solution of the DE $y' = y^2 + 1$ that satisfies $y(1) = 0$.

\[
\int \frac{1}{y^2+1} \, dy = \int 1 \, dx \Rightarrow \arctan(y) = x + C \Rightarrow y = \tan(x + C).
\]

If $y(1) = 0$ is to hold, we must have $0 = \tan(1 + C) \Rightarrow C = -1$. So the exact solution is $y = \tan(x - 1)$. Here is a picture of the direction field and the solution curve passing through $(1,0)$.
6. Find an equation of the tangent line to the curve 
\[ x = 3t^2 + 1, \quad y = 2t^3 + 1 \]
at the point (4,3). Is the particle ever motionless? Sketch the path of the particle.

We begin with a sketch of the curve parameterized by these equations.
The particle is at the point (4,3) when \( t = 1 \). The slope of the tangent line at (almost) any point is \( \frac{y}{x'}(t) = \frac{6t^2}{6t} = 1 \) when \( t = 1 \). Therefore, the tangent line is \( y - 3 = 1(x - 4) \Rightarrow y = x - 1 \).
The particle is instantaneously motionless if its speed is 0 at some point. The speed is given by \( \sqrt{(x'(t))^2 + (y'(t))^2} \).
The particle is instantaneously motionless if its speed is 0 at some point. The speed is given by
\[ \sqrt{(6 t)^2 + (6 t)^2} = 6 t \sqrt{1 + t^2} \]. This last expression is 0 only at \( t = 0 \). So the particle is instantaneously motionless at \( t = 0 \), which corresponds to the point (1,1) on the curve (the "cusp").

7. Write down, but do not evaluate, the integral giving the distance traveled by the particle in question 6 between \( t = 1 \) and \( t = 3 \).

\[ \int_{t=1}^{3} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \int_{1}^{3} \left( 6 t \sqrt{1 + t^2} \right) \, dt \]

8. Find the area of the region bounded by the x-axis and the curve given by \( x(t) = \ln(t) \), \( y(t) = t^2 - 4t + 3 \). Also, at what point is the tangent line horizontal?

Observe that \( y(t) = (t - 3)(t - 1) \) is 0 when \( t = 1, 3 \). Thus, the curve intersects the x-axis twice, when \( t = 1 \) at the point (0,0) and when \( t = 3 \) at the point (ln3, 0). Here is a sketch of the region:

\[ \text{In[123]} \quad \text{ParametricPlot} \left[ \{ \text{Log}[t], t^2 - 4t + 3 \}, \{ t, 1, 3 \} \right] \]

To find the area, we can use vertical strips of height \( 0 - y(t) = -(t^2 - 4t + 3) \) and thickness \( dx = \frac{dx}{dt} \, dt = \frac{1}{t} \, dt \). Therefore, the area of the region is

\[ \text{In[125]} \quad N \left[ \int_{1}^{3} \left( -(t^2 - 4t + 3) \left( \frac{1}{t} \right) \right) \, dt \right] \]

\[ \text{Out[125]} = 0.704163 \]
To determine where the slope of the tangent line is 0, we set \( \frac{y(t)}{x(t)} = 0 \Rightarrow y'(t) = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2 \). So, the tangent slope is 0 at the point corresponding to \( t = 2 \), namely, \((\ln(2), -1)\), which is the “vertex” of our “distorted parabola.”