Multinational Production, Risk Sharing, and Home Equity Bias

Technical Appendix

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1 Model Details

This Appendix shows derivations for Section 2.

1.1 Derivation of price indices, demand for goods, and real exchange rate

First, we derive the price index in the home country, P_t. It consists of the price index of goods produced by home firms in the home country, P_{Ht}, and price index of goods produced by foreign firms in the home country, P_{Ft}. C_t is the home consumer's consumption basket consisting of consumption of goods produced by home firms in the home country, C_{Ht}, and consumption of goods produced by foreign firms in the home country, C_{Ft}. ω is the parameter indicating the elasticity of substitution between home and foreign goods.

min $P_{Ht}C_{Ht} + P_{Ft}C_{Ft}$ subject to $C_t = 1$ where $C_t = [a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}}$

$\mathcal{L} = P_{Ht}C_{Ht} + P_{Ft}C_{Ft} - P_t[[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}} - 1]$

$\frac{\partial \mathcal{L}}{\partial C_{Ht}} = P_{Ht} - P_t \frac{\omega}{\omega-1}[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}-1} a^{\frac{1}{\omega}} \frac{\omega-1}{\omega} C_{Ht}^{\frac{\omega}{\omega-1}-1} = 0$

$P_{Ht} = P_t \frac{\omega}{\omega-1}[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}-1} a^{\frac{1}{\omega}} \frac{\omega-1}{\omega} C_{Ht}^{\frac{\omega}{\omega-1}-1}$

$\frac{\partial \mathcal{L}}{\partial C_{Ft}} = P_{Ft} - P_t \frac{\omega}{\omega-1}[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}-1} (1-a)^{\frac{1}{\omega}} \frac{\omega-1}{\omega} C_{Ft}^{\frac{\omega}{\omega-1}-1} = 0$

$P_{Ft} = P_t \frac{\omega}{\omega-1}[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}-1} (1-a)^{\frac{1}{\omega}} \frac{\omega-1}{\omega} C_{Ft}^{\frac{\omega}{\omega-1}-1}$

Since $[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}} \equiv C_t$ and $C_t = 1$, it is possible to write

$[a^{\frac{1}{\omega}} C_{Ht}^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} C_{Ft}^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}-1} = C_t^{\frac{1}{\omega}} = 1$

$P_{Ht} = P_t a^{\frac{1}{\omega}} C_{Ht}^{-\frac{1}{\omega}}$, so $C_{Ht}^{-\frac{1}{\omega}} = \frac{P_{Ht}}{P_t} a^{-\frac{1}{\omega}}$, so $C_{Ht} = (\frac{P_t}{P_{Ht}})^{\omega} a$ [1]

$P_{Ft} = P_t (1-a)^{\frac{1}{\omega}} C_{Ft}^{-\frac{1}{\omega}}$, so $C_{Ft}^{-\frac{1}{\omega}} = \frac{P_{Ft}}{P_t} (1-a)^{-\frac{1}{\omega}}$, so $C_{Ft} = (\frac{P_t}{P_{Ft}})^{\omega}(1-a)$

Substitute into $C_t = 1$

$[a^{\frac{1}{\omega}} (\frac{P_t}{P_{Ht}})^{\omega-1} a^{\frac{\omega-1}{\omega}} + (1-a)^{\frac{1}{\omega}} (\frac{P_t}{P_{Ft}})^{\omega-1} (1-a)^{\frac{\omega-1}{\omega}}]^{\frac{\omega}{\omega-1}} = 1$

$a^{\frac{\omega}{\omega-1}} (\frac{P_t}{P_{Ht}})^{\omega} + (1-a)^{\frac{\omega}{\omega-1}} (\frac{P_t}{P_{Ft}})^{\omega} = 1$

$a(\frac{P_t}{P_{Ht}})^{\omega-1} + (1-a)(\frac{P_t}{P_{Ft}})^{\omega-1} = 1$

$\frac{aP_{Ht}^{1-\omega} + (1-a)P_{Ft}^{1-\omega}}{P_t^{1-\omega}} = 1$

$P_t = [aP_{Ht}^{1-\omega} + (1-a)P_{Ft}^{1-\omega}]^{\frac{1}{1-\omega}}$, which is the price index in the home country.

[1] Note that this expression should be completely written as $C_{Ht} = (\frac{P_t}{P_{Ht}})^{\omega} a C_t$ but we drop C_t because we imposed $C_t = 1$.

Now, we derive the price index of goods produced by home firms in the home country, P_{Ht}. In this derivation, the home consumer's consumption basket, C_{Ht}, consists of goods produced by the home firms z where we integrate from 0 to a because there are a home firms:

$\min p_t(z)c_t(z)$ subject to $C_{Ht} = 1$ where $C_{Ht} = [(\frac{1}{a})^{\frac{1}{\theta}} \int_0^a c_t(z)^{\frac{\theta-1}{\theta}} dz]^{\frac{\theta}{\theta-1}}$

$\mathcal{L} = p_t(z)c_t(z) - P_{Ht}[[(\frac{1}{a})^{\frac{1}{\theta}} \int_o^a c_t(z)^{\frac{\theta-1}{\theta}} dz]^{\frac{\theta}{\theta-1}} - 1]$

$\frac{\partial \mathcal{L}}{\partial c_t(z)} = p_t(z) - P_{Ht}\frac{\theta}{\theta-1}[(\frac{1}{a})^{\frac{1}{\theta}} \int_o^a c_t(z)^{\frac{\theta-1}{\theta}} dz]^{\frac{\theta}{\theta-1}-1}(\frac{1}{a})^{\frac{1}{\theta}}\frac{\theta-1}{\theta}c_t(z)^{\frac{\theta-1}{\theta}-1} = 0$

$p_t(z) = P_{Ht}[(\frac{1}{a})^{\frac{1}{\theta}} \int_o^a c_t(z)^{\frac{\theta-1}{\theta}} dz]^{\frac{1}{\theta-1}}(\frac{1}{a})^{\frac{1}{\theta}}c_t(z)^{-\frac{1}{\theta}}$

$c_t(z)^{-\frac{1}{\theta}} = \frac{p_t(z)}{P_{Ht}}a^{\frac{1}{\theta}}$

$c_t(z) = \frac{1}{a}(\frac{p_t(z)}{P_{Ht}})^{-\theta}$[2]

Substitute this expression into $C_{Ht} = 1$:

$[(\frac{1}{a})^{\frac{1}{\theta}} \int_o^a (\frac{P_{Ht}}{p_t(z)})^{\theta-1}(\frac{1}{a})^{\frac{\theta-1}{\theta}} dz]^{\frac{\theta}{\theta-1}} = 1$

$[(\frac{1}{a})^{\frac{1}{\theta}}(\frac{1}{a})^{\frac{\theta-1}{\theta}} \int_o^a (\frac{P_{Ht}}{p_t(z)})^{\theta-1} dz]^{\frac{\theta}{\theta-1}} = 1$

$P_{Ht}^{\theta}[\frac{1}{a} \int_o^a (\frac{1}{p_t(z)})^{\theta-1} dz]^{\frac{\theta}{\theta-1}} = 1$

$[\frac{1}{a} \int_o^a p_t(z)^{1-\theta} dz]^{\frac{\theta}{\theta-1}} = P_{Ht}^{-\theta}$

$[\frac{1}{a} \int_o^a p_t(z)^{1-\theta} dz]^{\frac{-1}{\theta-1}} = P_{Ht}$

$[\frac{1}{a} \int_o^a p_t(z)^{1-\theta} dz]^{\frac{1}{1-\theta}} = P_{Ht}$, which is the price index of goods produced by home firms (denoted by z) in the home country.

We can then write the demand for home firm z output by the representative household in the home country based on the above as:

$c_t(z) = \frac{1}{a}(\frac{p_t(z)}{P_{Ht}})^{-\theta}C_{Ht} = \frac{1}{a}(\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}aC_t = (\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}C_t$[3].

Since there are a home households, the demand for home firm z output by all households in the home country is: $(\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}aC_t$.

[2]Note that this expression should be completely written as $c_t(z) = \frac{1}{a}(\frac{P_{Ht}}{p_t(z)})^{\theta}C_{Ht}$ but we drop C_{Ht} because we imposed $C_{Ht} = 1$.

[3]Note that in this expression we should write $(C_t + G_t)$ to reflect the total demand made by the home country that comes from home consumers as well as home government. However, for the purpose of this derivation, we can omit G_t.

The demand for home firm z output by all households and government in the home country is: $(\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}(aC_t + aG_t)$ assuming that the government spends G_t per capita. Notice: $a(C_t + G_t)$ is Y_t^d, i.e, demand for consumption basket in the home country. Note that in contrast to Ghironi, Lee, and Rebucci (2015) (GLR), we do not have Y_t^W. Note: The total per capita demand for consumption basket in the home country is: $y_t^d = C_t + G_t$

The price index of goods produced by foreign firms in the home country can be derived by following the same steps. In this derivation, the home consumer's consumption basket, C_{Ft}, consists of goods produced by the foreign firms z^* where we integrate from a to $1-a$ because there are $1-a$ foreign firms:
$[\frac{1}{1-a}\int_a^1 p_t(z^*)^{1-\theta}dz^*]^{\frac{1}{1-\theta}} = P_{Ft}$ using consumption of goods produced by foreign firms in the home country, $C_{Ft} = [(\frac{1}{1-a})^{\frac{1}{\theta}}\int_a^1 c_t(z^*)^{\frac{\theta-1}{\theta}}dz^*]^{\frac{\theta}{\theta-1}}$

The derivation of the price index of goods produced in the foreign country (consisting of a price index of goods produced by home firms in the foreign country, P^*_{Ht}, and a price index of goods produced by foreign firms in the foreign country, P^*_{Ft}), P^*_t, yields:
$P^*_t = [aP^{*1-\omega}_{Ht} + (1-a)P^{*1-\omega}_{Ft}]^{\frac{1}{1-\omega}}$

Note that the expressions for P_{Ht}, P_{Ft}, P^*_{Ht} and P^*_{Ft} (and, hence, P_t and P^*_t) are identical to GLR. However, since purchasing power parity does not hold in our model, we have to take the real exchange rate, Q_t, into account.
$Q_t \equiv \frac{\varepsilon_t P^*_t}{P_t}$ where ε_t is the nominal exchange rate, and $\varepsilon_t P^*_t = [a(\varepsilon_t P^*_{Ht})^{1-\omega} + (1-a)(\varepsilon_t P^*_{Ft})^{1-\omega}]^{\frac{1}{1-\omega}}$.
Then: $Q_t = [\frac{a(\varepsilon_t P^*_{Ht})^{1-\omega} + (1-a)(\varepsilon_t P^*_{Ft})^{1-\omega}}{aP^{1-\omega}_{Ht} + (1-a)P^{1-\omega}_{Ft}}]^{\frac{1}{1-\omega}}$

1.2 Household optimization

Start with the home household budget constraint in nominal terms in home currency:

$$(V_t + D_t + \varepsilon_t D^*_t)x_t + (\varepsilon_t V^*_t + D_{*t} + \varepsilon_t D^*_{*t})x^*_t + W_t L_t = V_t x_{t+1} + \varepsilon_t V^*_t x^*_{t+1} + P_t C_t + P_t G_t,$$

where x_t denotes shares of the home firm, x^*_t denotes shares of the foreign firm, V_t is the price of the home firm's shares, V^*_t is the price of the foreign firm's shares, D_t is the dividend of the home firm in the home country, D_t^* is the dividend of the home firm in the foreign country, D_{*t}^* is the dividend of the foreign firm in the foreign country, and D_{*t} is the dividend of the foreign firm in the home country.

Divide by P_t to convert into units of home country's consumption basket:

$$(v_t + d_t + d_t^*)x_t + (v_t^* + d_{*t} + d_{*t}^*)x_t^* + w_t L_t = v_t x_{t+1} + v_t^* x_{t+1}^* + C_t + G_t$$

In this notation, large case letters are used for nominal variables, and small case letters are used for real variables. For example, W_t is nominal wage, and w_t is real wage.

$\mathcal{L} = E_t \sum_{s=t}^{\infty} \beta^{s-t} (\frac{C_s^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \chi \frac{L_s^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}) +$
$+ \lambda_t[(v_t + d_t + d_t^*)x_t + (v_t^* + d_{*t} + d_{*t}^*)x_t^* + w_t L_t - v_t x_{t+1} - v_t^* x_{t+1}^* - C_t - G_t]$
This Lagrangian is identical to the GLR Lagrangian except for the additional terms to account for the dividends coming from two different countries. φ is the parameter indicating elasticity of labor with $\varphi = 0$ denoting inelastic labor and $\varphi > 1$ denoting elastic labor.

First order conditions (FOCs):

With respect to C_t:
$\frac{\partial \mathcal{L}}{\partial C_t} = (1 - \frac{1}{\sigma}) \frac{C_t^{\frac{\sigma-1}{\sigma}-1}}{1-\frac{1}{\sigma}} + \lambda_t(-1) = 0$
$C_t^{-\frac{1}{\sigma}} = \lambda_t,$
which is the same as in GLR.

With respect to L_t:
$\frac{\partial \mathcal{L}}{\partial L_t} = -\chi(1 + \frac{1}{\varphi}) \frac{L_t^{\frac{\varphi+1}{\varphi}-1}}{1+\frac{1}{\varphi}} + \lambda_t w_t = 0$
$\chi L_t^{\frac{\varphi+1-\varphi}{\varphi}} = \lambda_t w_t$
$\chi L_t^{\frac{1}{\varphi}} = C_t^{-\frac{1}{\sigma}} w_t$
$L_t^{\frac{1}{\varphi}} = \frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi}$
$L_t = (\frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi})^{\varphi},$
which is the same as in GLR.

With respect to x_{t+1}:

$\frac{\partial \mathcal{L}}{\partial x_{t+1}} = \lambda_t(-v_t) + \beta E_t\{\lambda_{t+1}(v_{t+1} + d_{t+1} + d^*_{t+1})\} = 0$

$C_t^{-\frac{1}{\sigma}} v_t = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}(v_{t+1} + d_{t+1} + d^*_{t+1})\}$

$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} \frac{v_{t+1}+d_{t+1}+d^*_{t+1}}{v_t}\} \equiv \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} R_{t+1}\}$

where the definition of R_{t+1} differs from GLR due to the home firm's dividends coming from two different countries.

With respect to x^*_{t+1}:

$\frac{\partial \mathcal{L}}{\partial x^*_{t+1}} = \lambda_t(-v^*_t) + \beta E_t\{\lambda_{t+1}(v^*_{t+1} + d_{*t+1} + d^*_{*t+1})\} = 0$

$C_t^{-\frac{1}{\sigma}} v^*_t = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}(v^*_{t+1} + d_{*t+1} + d^*_{*t+1})\}$

$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} \frac{v^*_{t+1}+d_{*t+1}+d^*_{*t+1}}{v^*_t}\} \equiv \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} R^*_{t+1}\}$

where the definition of R^*_{t+1} differs from GLR due to the foreign firm's dividends coming from two different countries.

1.3 Derivation of optimal labor demands and prices:

We set up the firm's problem as profit maximization. The revenue of the home firm, z, consists of revenue earned in the home country and revenue earned in the foreign country. The revenue earned in the home country is $p_t(z)Z_tL_t(z)$ because the home firm employs home labor, $L_t(z)$, and applies home productivity, Z_t, to produce its output in the home country. This output is then multiplied by the price charged by the home firm in the home country, $p_t(z)$. This is stated in home currency. The revenue earned in the foreign country is $p^*_t(z)Z_t^{\gamma}Z_t^{*1-\gamma}L^*_t(z)$ because the firm employs foreign labor, $L^*_t(z)$, and applies productivity $Z_t^{\gamma}Z_t^{*1-\gamma}$. This output is then multiplied by the price charged by the home firm in the foreign country, $p^*_t(z)$. Since $p^*_t(z)Z_t^{\gamma}Z_t^{*1-\gamma}L^*_t(z)$ is stated in foreign currency, we multiply it by the nominal exchange rate, ε_t, to convert it to the home currency. The cost of the home firm consists of the labor cost incurred in the home country, $W_tL_t(z)$, and the cost incurred in the foreign country, $W^*_tL^*_t(z)$ that again has to be multiplied by the nominal exchange rate to convert to home currency. The problem, therefore, becomes:

$$\text{Max } p_t(z)Z_tL_t(z) + \varepsilon_t p^*_t(z)Z_t^{\gamma}Z_t^{*1-\gamma}L^*_t(z) - W_tL_t(z) - \varepsilon_t W^*_t L^*_t(z)$$

subject to:

$Y_t^s(z) = Y_t^d(z)$, which says that output supplied by the home firm in the home country has to equal this firm's output demanded in the home country,

and

$Y_t^{s*}(z) = Y_t^{d*}(z)$, which says that output supplied by the home firm in the foreign country has to equal this firm's output demanded in the foreign country.

To derive the optimal demand for labor by home firm, z, in the home country, we use $Y_t^s(z) = Y_t^d(z)$. $Y_t^s(z)$ comes from the production function, i.e., $Y_t^s(z) = Z_t L_t(z)$. $Y_t^d(z)$ comes from the demand for home firm's z good that was derived in Section 1.1: $Y_t^d(z) = (\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}(aC_t + aG_t)$ (which is $Y_t^d(z) = (\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega} Y_t^d$ because $(aC_t + aG_t)$ represents the demand by all home households and government). Both the $Y_t^d(z)$ and $Y_t^s(z)$ are in units of the home consumption.

$$Z_t L_t(z) = (\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}(aC_t + aG_t)$$
$$L_t(z) = (\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z_t}$$

To derive the optimal demand for labor by the home firm, z, in the foreign country, we use $Y_t^{*s}(z) = Y_t^{*d}(z)$. $Y_t^{*s}(z)$ comes from the production function, i.e., $Y_t^{*s}(z) = Z_t^{\gamma} Z_t^{*1-\gamma} L_t^*(z)$. $Y_t^{*d}(z)$ comes from the demand for firm z, i.e., $Y_t^{*d}(z) = (\frac{p_t^*(z)}{P_{Ht}^*})^{-\theta}(\frac{P_{Ht}^*}{P_t^*})^{-\omega}((1-a)C_t^* + (1-a)G_t^*)$ where the $(1-a)$ is included in this expression because there are $(1-a)$ households in the foreign country and we are assuming that the foreign government spends G_t^* per capita.

$$Z_t^{\gamma} Z_t^{*1-\gamma} L_t^*(z) = (\frac{p_t^*(z)}{P_{Ht}^*})^{-\theta}(\frac{P_{Ht}^*}{P_t^*})^{-\omega}((1-a)C_t^* + (1-a)G_t^*)$$
$$L_t^*(z) = (\frac{p_t^*(z)}{P_{Ht}^*})^{-\theta}(\frac{P_{Ht}^*}{P_t^*})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma} Z_t^{*1-\gamma}}$$

The foreign firm's problem is:

$$\text{Max } p_{*t}(z^*) Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t}(z^*) + \varepsilon_t p_{*t}^*(z^*) Z_t^* L_{*t}^*(z^*) - W_t L_{*t}(z^*) - \varepsilon_t W_t^* L_{*t}^*(z^*)$$

where $p_{*t}(z^*)$ is price charged by the foreign firm, z^*, in the home country,

$p_{*t}^*(z^*)$ is price charged by the foreign firm, z^*, in the foreign country,

$L_{*t}(z^*)$ is labor employed by the foreign firm, z^*, in the home country, and

$L_{*t}^*(z^*)$ is labor employed by the foreign firm, z^*, in the foreign country subject to:

$Y^s_{*t}(z^*) = Y^d_{t*}(z^*)$, which says that output supplied by the foreign firm in the home country has to equal this firm's output demanded in the home country,

and

$Y^{*s}_{*t}(z^*) = Y^{*d}_{t*}(z^*)$, which says that output supplied by the foreign firm in the foreign country has to equal this firm's output demanded in the foreign country.

To derive the optimal demand for labor by the foreign firm, z^*, in the home country, we use $Y^s_{*t}(z^*) = Y^d_{t*}(z^*)$. $Y^s_{*t}(z^*)$ comes from the production function, i.e., $Y^s_{*t}(z^*) = Z_t^{1-\gamma} Z_t^{*\gamma} L_{t*}(z^*)$. $Y^d_{t*}(z^*)$ comes from the demand for firm z^* good: $Y^d_{t*}(z^*) = (\frac{p_{*t}(z^*)}{P_{Ft}})^{-\theta}(\frac{P_{Ft}}{P_t})^{-\omega}(aC_t + aG_t)$.

$$Z_t^{1-\gamma} Z_t^{*\gamma} L_{t*}(z^*) = (\frac{p_{*t}(z^*)}{P_{Ft}})^{-\theta}(\frac{P_{Ft}}{P_t})^{-\omega}(aC_t + aG_t)$$
$$L_{t*}(z^*) = (\frac{p_{*t}(z^*)}{P_{Ft}})^{-\theta}(\frac{P_{Ft}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma} Z_t^{*\gamma}}$$

To derive the optimal demand for labor by foreign firm, z^*, in the foreign country, we use $Y^{*s}_{*t}(z^*) = Y^{*d}_{t*}(z^*)$. $Y^{*s}_{*t}(z^*)$ comes from the production function, i.e., $Y^{*s}_{*t}(z^*) = Z_t^* L^*_{*t}(z^*)$. $Y^{*d}_{t*}(z^*)$ comes from the demand for firm z^* good: $Y^{*d}_{t*}(z^*) = (\frac{p_{*t}(z^*)}{P^*_{Ft}})^{-\theta}(\frac{P^*_{Ft}}{P_t})^{-\omega}((1-a)C_t^* + (1-a)G_t^*))$.

$$Z_t^* L^*_{*t}(z^*) = (\frac{p_{*t}(z^*)}{P^*_{Ft}})^{-\theta}(\frac{P^*_{Ft}}{P_t})^{-\omega}((1-a)C_t^* + (1-a)G_t^*))$$
$$L^*_{*t}(z^*) = (\frac{p_{*t}(z^*)}{P^*_{Ft}})^{-\theta}(\frac{P^*_{Ft}}{P_t})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$$

To derive prices, we can substitute these labor demands in the maximization problems:
For the home firm, z, the problem becomes:

$$\text{Max } p_t(z) Z_t(\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z_t} + \varepsilon_t p_t^*(z) Z_t^{\gamma} Z_t^{*1-\gamma}(\frac{p_t^*(z)}{P^*_{Ht}})^{-\theta}(\frac{P^*_{Ht}}{P^*_t})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma} Z_t^{*1-\gamma}}$$
$$-W_t(\frac{p_t(z)}{P_{Ht}})^{-\theta}(\frac{P_{Ht}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z_t} - \varepsilon_t W_t^*(\frac{p_t^*(z)}{P^*_{Ht}})^{-\theta}(\frac{P^*_{Ht}}{P^*_t})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma} Z_t^{*1-\gamma}}$$

Take the derivative with respect to $p_t(z)$:
$\theta - 1 = \frac{\theta}{p_t(z)}\frac{W_t}{Z_t}$
$p_t(z) = \frac{\theta}{\theta-1}\frac{W_t}{Z_t}$, which is the price charged by the home firm in the home country.
Take the derivative with respect to $p_t^*(z)$:
$\theta - 1 = \frac{\theta}{p_t^*(z)}\frac{W_t^*}{Z_t^{\gamma} Z_t^{*1-\gamma}}$

$p^*_t(z) = \frac{\theta}{\theta-1}\frac{W^*_t}{Z^\gamma_t Z^{*1-\gamma}_t}$, which is the price charged by the home firm in the foreign country.

For the foreign firm, z^*, the problem becomes:

$$\text{Max } p_{*t}(z^*)Z^{1-\gamma}_t Z^{*\gamma}_t(\frac{p_{*t}(z^*)}{P_{Ft}})^{-\theta}(\frac{P_{Ft}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z^{1-\gamma}_t Z^{*\gamma}_t} + \varepsilon_t p^*_{*t}(z^*)Z^*_t(\frac{p_{*t}(z^*)}{P^*_{Ft}})^{-\theta}(\frac{P^*_{Ft}}{P_t})^{-\omega}\frac{(1-a)(C^*_t+G^*_t)}{Z^*_t}$$
$$-W_t(\frac{p_{*t}(z^*)}{P_{Ft}})^{-\theta}(\frac{P_{Ft}}{P_t})^{-\omega}\frac{a(C_t+G_t)}{Z^{1-\gamma}_t Z^{*\gamma}_t} - \varepsilon_t W^*_t(\frac{p_{*t}(z^*)}{P^*_{Ft}})^{-\theta}(\frac{P^*_{Ft}}{P_t})^{-\omega}\frac{(1-a)(C^*_t+G^*_t)}{Z^*_t}$$

Take the derivative with respect to $p_{*t}(z^*)$:

$(1-\theta) = \frac{\theta W_t}{Z^{1-\gamma}_t Z^{*\gamma}_t p_{*t}(z^*)}$

$p_{*t}(z^*) = \frac{\theta}{\theta-1}\frac{W_t}{Z^{1-\gamma}_t Z^{*\gamma}_t}$, which is the price charged by the foreign firm in the home country.

Take the derivative with respect to $p^*_{*t}(z^*)$:

$(1-\theta) = \frac{\theta W^*_t}{Z^*_t p^*_{*t}(z^*)}$

$p^*_{*t}(z^*) = \frac{\theta}{\theta-1}\frac{W^*_t}{Z^*_t}$, which is the price charged by the foreign firm in the foreign country.

In equilibrium, $p_t(z) = P_{Ht}$, which says that price charged by home firm z in home country equals the price index for goods produced by home firms. Similarly, $p^*_t(z) = P^*_{Ht}$ for price charged by home firms in the foreign country, $p_{*t}(z^*) = P_{Ft}$ for price charged by foreign firms in the home country, and $p^*_{*t}(z^*) = P^*_{Ft}$ for price charged by foreign firms in the foreign country.

Therefore:

$P_{Ht} = \frac{\theta}{\theta-1}\frac{W_t}{Z_t}$ for price index of goods produced by home firms in the home country,

$P^*_{Ht} = \frac{\theta}{\theta-1}\frac{W^*_t}{Z^\gamma_t Z^{*1-\gamma}_t}$ for price index of goods produced by home firms in the foreign country,

$P_{Ft} = \frac{\theta}{\theta-1}\frac{W_t}{Z^{1-\gamma}_t Z^{*\gamma}_t}$ for price index of goods produced by foreign firms in the home country, and

$P^*_{Ft} = \frac{\theta}{\theta-1}\frac{W^*_t}{Z^*_t}$ for price index of goods produced by foreign firms in the foreign country.

Then, we can write expressions for relative prices:

$RP_t = \frac{p_t(z)}{P_t} = \frac{P_{Ht}}{P_t} = \frac{\theta}{\theta-1}\frac{w_t}{Z_t}$ for price charged by a home firm in the home country relative to the home country's price level in units of the home country consumption,

$RP^*_t = \frac{p^*_t(z)}{P^*_t} = \frac{P^*_{Ht}}{P^*_t} = \frac{\theta}{\theta-1}\frac{w^*_t}{Z^\gamma_t Z^{*1-\gamma}_t}$ for price charged by a home firm in the foreign country relative to the foreign country's price level in units of the foreign country consumption, $RP_{*t} = \frac{p_{*t}(z^*)}{P_t} = \frac{P_{Ft}}{P_t} = \frac{\theta}{\theta-1} \frac{w_t}{Z_t^{1-\gamma} Z_t^{*\gamma}}$ for price charged by a foreign firm in the home country relative to the home country's price level in units of the home country consumption, and $RP_{*t}^* = \frac{p_{*t}^*(z^*)}{P_t^*} = \frac{P_{Ft}^*}{P_t^*} = \frac{\theta}{\theta-1} \frac{w_t^*}{Z_t^*}$ for price charged by a foreign firm in the foreign country relative to the foreign country's price level in units of the foreign country consumption. Note that the small case letter, w, is used to denote real wage as opposed to the large case letter W that denotes nominal wage.

The optimal labor demands can be rewritten with relative prices as:

Optimal demand for labor by a home firm, z, in the home country becomes:

$L_t(z) = RP_t^{-\omega} \frac{a(C_t+G_t)}{Z_t}$ using $p_t(z) = P_{Ht}$ in equilibrium and $RP_t = \frac{P_{Ht}}{P_t}$

Optimal demand for labor by a home firm, z, in the foreign country becomes:

$L_t^*(z) = RP_t^{*-\omega} \frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma} Z_t^{*1-\gamma}}$ using $p_t^*(z) = P_{Ht}^*$ in equilibrium and $RP_t^* = \frac{P_{Ht}^*}{P_t^*}$

Optimal demand for labor by a foreign firm, z^*, in the home country becomes:

$L_{t*}(z^*) = RP_{*t}^{-\omega} \frac{a(C_t+G_t)}{Z_t^{1-\gamma} Z_t^{*\gamma}}$ using $p_{*t}(z^*) = P_{Ft}$ in equilibrium and $RP_{*t} = \frac{P_{Ft}}{P_t}$

Optimal demand for labor by a foreign firm, z^*, in the foreign country becomes:

$L_{*t}^*(z^*) = RP_{*t}^{*-\omega} \frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$ using $p_{*t}^*(z^*) = P_{Ft}^*$ in equilibrium and $RP_{*t}^* = \frac{P_{Ft}^*}{P_t^*}$

We need to account for the number of firms in each country.

There are a home firms in the home country, so the optimal demand for labor by all home firms in the home country is:

$$aL_t(z) = aRP_t^{-\omega} \frac{a(C_t+G_t)}{Z_t}$$

Total per capita labor demand by all home firms in the home country is:

$$\frac{aL_t(z)}{a} = \frac{aRP_t^{-\omega} \frac{a(C_t+G_t)}{Z_t}}{a}$$

$$\frac{a}{a} L_t(z) = \frac{a}{a} RP_t^{-\omega} \frac{a(C_t+G_t)}{Z_t}$$

$$L_t(z) = RP_t^{-\omega} \frac{a(C_t+G_t)}{Z_t}$$

where we divide by a because there are a households in the home country.

There are $(1-a)$ foreign firms in the home country, so the optimal demand for labor by all foreign firms in the home country is:

$$(1-a)L_{t*}(z^*) = (1-a)RP_{*t}^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma}Z_t^{*\gamma}}$$

Per capita labor demand by all foreign firms in home country is:

$$\frac{1-a}{a}L_{t*}(z^*) = \frac{1-a}{a}RP_{*t}^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma}Z_t^{*\gamma}}$$

where we again divide by a because there are a households in the home country.
There are a home firms in the foreign country, so the optimal demand for labor by all home firms in the foreign country is:

$$aL_t^*(z) = aRP_t^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma}Z_t^{*1-\gamma}}$$

Per capita labor demand by all home firms in foreign country is:

$$\frac{a}{1-a}L_t^*(z) = \frac{a}{1-a}RP_t^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma}Z_t^{*1-\gamma}}$$

where we divide by $(1-a)$ because there are $(1-a)$ households in the home country.
There are $(1-a)$ foreign firms in the foreign country, so the optimal demand for labor by all foreign firms in the foreign country is:

$$(1-a)L_{*t}^*(z^*) = (1-a)RP_{*t}^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$$

Total per capita labor demand by all foreign firms in the foreign country is:

$$\frac{1-a}{1-a}L_{*t}^*(z^*) = \frac{1-a}{1-a}RP_{*t}^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$$

where we again divide by $(1-a)$ because there are $(1-a)$ households in the home country.

1.4 Net foreign assets (NFA) law of motion

Start with the home household budget constraint in units of the home country's consumption basket from Section 1.2:

$$(v_t+d_t+d_t^*)x_t + (v_t^*+d_{*t}+d_{*t}^*)x_t^* + w_tL_t = v_tx_{t+1} + v_t^*x_{t+1}^* + C_t + G_t$$

Then:

$$(v_t + d_t + d_t^*)x_t + (v_t^* + d_{*t} + d_{*t}^*)x_t^* + w_t L_t = v_t x_{t+1} + nfa_{t+1} + \frac{1-a}{a} v_t x_{*t+1} + C_t + G_t$$

where net foreign assets, nfa_{t+1}, is defined as $nfa_{t+1} \equiv v_t^* x_{t+1}^* - \frac{1-a}{a} v_t x_{*t+1}$, i.e., the value of home holdings of foreign shares minus the value of foreign holdings of home shares adjusted for population sizes of home and foreign countries, i.e., a and $1-a$, respectively, as in GLR. We defined return on holding home equity as $R_t \equiv \frac{v_t+d_t+d_t^*}{v_{t-1}}$ and return on holding foreign equity as $R_t^* \equiv \frac{v_t^*+d_{*t}+d_{*t}^*}{v_{t-1}^*}$ in Section 1.2, so:

$v_t x_{t+1} + nfa_{t+1} + \frac{1-a}{a} v_t x_{*t+1} + C_t + G_t = \frac{(v_t+d_t+d_t^*)v_{t-1}}{v_{t-1}} x_t + \frac{(v_t^*+d_{*t}+d_{*t}^*)v_{t-1}^*}{v_{t-1}^*} x_t^* + w_t L_t$

$v_t x_{t+1} + nfa_{t+1} + \frac{1-a}{a} v_t x_{*t+1} + C_t + G_t = R_t v_{t-1} x_t + R_t^* v_{t-1}^* x_t^* + w_t L_t$

$nfa_{t+1} = -v_t x_{t+1} - \frac{1-a}{a} v_t x_{*t+1} + R_t v_{t-1} x_t + R_t^* v_{t-1}^* x_t^* + w_t L_t - C_t - G_t$

$nfa_{t+1} = -v_t(x_{t+1} + \frac{1-a}{a} x_{*t+1}) + R_t v_{t-1} x_t + R_t^* v_{t-1}^* x_t^* + w_t L_t - C_t - G_t$

$nfa_{t+1} = -v_t + R_t v_{t-1} x_t + R_t^* v_{t-1}^* x_t^* + w_t L_t - C_t - G_t$

where market clearing condition $a x_{t+1} + (1-a) x_{*t+1} = a$ was used to obtain $x_{t+1} = 1 - \frac{1-a}{a} x_{*t+1}$ as in GLR.

$nfa_{t+1} = -v_t + R_t^* v_{t-1}^* x_t^* + R_t v_{t-1}(1 - \frac{1-a}{a} x_{*t}) + w_t L_t - C_t - G_t$ where we used $x_t = 1 - x_{*t}\frac{1-a}{a}$.

$nfa_{t+1} = -v_t + R_t^* v_{t-1}^* x_t^* + R_t v_{t-1} - R_t v_{t-1} \frac{1-a}{a} x_{*t} + w_t L_t - C_t - G_t$

$nfa_{t+1} = -v_t + R_t^* v_{t-1}^* x_t^* + v_t + d_t + d_t^* - R_t v_{t-1} \frac{1-a}{a} x_{*t} + w_t L_t - C_t - G_t$

$nfa_{t+1} = R_t^* v_{t-1}^* x_t^* - R_t v_{t-1} \frac{1-a}{a} x_{*t} + y_t - C_t - G_t$

where $y_t \equiv d_t + d_t^* + w_t L_t$, which differs from GLR due to the additional term d_t^*. Note that we assume that the dividend of the home firm producing in the foreign country, d_t^*, is a part of the home country GDP, i.e., we assume that firms repatriate profits to their countries of origin for distribution to domestic and foreign shareholders.

$nfa_{t+1} = R_t v_{t-1}^* x_t^* - R_t v_{t-1}^* x_t^* + R_t^* v_{t-1}^* x_t^* - R_t v_{t-1} \frac{1-a}{a} x_{*t} + y_t - C_t - G_t$

Define excess return from holding foreign equity $R_t^D = R_t^* - R_t$ and portfolio holding

$\alpha_t = v_{t-1}^* x_t^*$:

$nfa_{t+1} = R_t^D \alpha_t + R_t v_{t-1}^* x_t^* - R_t v_{t-1} \frac{1-a}{a} x_{*t} + y_t - C_t - G_t$

$nfa_{t+1} = R_t^D \alpha_t + R_t nfa_t + y_t - C_t - G_t$

where definition $nfa_t \equiv v_{t-1}^* x_t^* - \frac{1-a}{a} v_{t-1} x_{*t}$ was used.

This is identical to GLR except the definitions of R_t and R_t^*, and hence R_t^D, differ as explained above. This is in units of home consumption.

Similar derivations can be done to obtain the NFA law of motion for the foreign household:
$nfa^{*f}_{t+1} = R^{Df}_t \alpha^{*f}_t + R^f_t nfa^{*f}_t + y^{*f}_t - C^{*f}_t - G^{*f}_t$
This is in units of foreign consumption (denoted by the superscript f), and $R^f_t \equiv \frac{v^f_t + d^f_t + d^{f*}_t}{v^f_{t-1}}$, which is the foreign household's return on holding home firm's shares shown in units of foreign consumption.
To convert to units of home consumption:
$Q_t nfa^{*f}_{t+1} = \frac{Q_t}{Q_{t-1}} R^{Df}_t Q_{t-1} \alpha^{*f}_t + \frac{Q_t}{Q_{t-1}} R^f_t Q_{t-1} nfa^{*f}_t + Q_t y^{*f}_t - Q_t C^{*f}_t - Q_t G^{*f}_t$

Subtracting the home and foreign NFA laws of motions and using the superscript D to denote the difference between home and foreign variables gives:
$nfa_{t+1} - Q_t nfa^{*f}_{t+1} = R^D_t \alpha_t + R_t nfa_t + y_t - C_t - G_t - [\frac{Q_t}{Q_{t-1}} R^{Df}_t Q_{t-1} \alpha^{*f}_t + \frac{Q_t}{Q_{t-1}} R^f_t Q_{t-1} nfa^{*f}_t + Q_t y^{*f}_t - Q_t C^{*f}_t - Q_t G^{*f}_t]$
$nfa_{t+1} - Q_t nfa^{*f}_{t+1} = (R^D_t \alpha_t - \frac{Q_t}{Q_{t-1}} R^{Df}_t Q_{t-1} \alpha^{*f}_t) + (R_t nfa_t - \frac{Q_t}{Q_{t-1}} R^f_t Q_{t-1} nfa^{*f}_t) + (y_t - Q_t y^{*f}_t) - (C_t - Q_t C^{*f}_t) - (G_t - Q_t G^{*f}_t)$
We use the following clearing conditions:
For net foreign assets: $a nfa_t + (1-a) Q_{t-1} nfa^{*f}_t = 0$, which gives $\frac{-a}{1-a} \frac{1}{Q_{t-1}} nfa_t = nfa^{*f}_t$
For portfolios: $a\alpha_t + (1-a)\alpha^{*f}_t Q_{t-1} = 0$, which gives $\alpha^{*f}_t = \frac{-a}{1-a} \frac{1}{Q_t - 1} \alpha_t$
LHS becomes: $nfa_{t+1} - Q_t[\frac{-a}{1-a} \frac{1}{Q_t} nfa_{t+1}] = nfa_{t+1}(1 + \frac{a}{1-a}) = nfa_{t+1} \frac{1}{1-a}$
RHS becomes: $(R^D_t \alpha_t - \frac{Q_t}{Q_{t-1}} R^{Df}_t Q_{t-1} \frac{-a}{1-a} \frac{1}{Q_{t-1}} \alpha_t) + (R_t nfa_t - \frac{Q_t}{Q_{t-1}} R^f_t Q_{t-1} \frac{-a}{1-a} \frac{1}{Q_{t-1}} nfa_t) + (y_t - Q_t y^{*f}_t) - (C_t - Q_t C^{*f}_t) - (G_t - Q_t G^{*f}_t)$
Use: $R^D_t = \frac{Q_t}{Q_{t-1}} R^{Df}_t$ and $R_t = \frac{Q_t}{Q_{t-1}} R^f_t$
RHS becomes: $(R^D_t \alpha_t - R^D_t \frac{-a}{1-a} \alpha_t) + (R_t nfa_t - R_t \frac{-a}{1-a} nfa_t) + (y_t - Q_t y^{*f}_t) - (C_t - Q_t C^{*f}_t) - (G_t - Q_t G^{*f}_t)$

Then:
$nfa_{t+1} \frac{1}{1-a} = R^D_t \alpha_t \frac{1}{1-a} + R_t nfa_t \frac{1}{1-a} + (y_t - Q_t y^{*f}_t) - (C_t - Q_t C^{*f}_t) - (G_t - Q_t G^{*f}_t)$
$nfa_{t+1} = R^D_t \alpha_t + R_t nfa_t + (1-a)[(y_t - Q_t y^{*f}_t) - (C_t - Q_t C^{*f}_t) - (G_t - Q_t G^{*f}_t)]$

1.5 Expression for relative GDP

In this section, we derive an expression for the relative GDP, $\frac{y_t}{y^*_t}$.

Derivation of home GDP, y_t, i.e., output produced by home and foreign firms in the home country:

$y_t = RP_t Z_t L_t + RP_{*t} Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t} = \frac{\theta}{\theta-1}\frac{w_t}{Z_t} Z_t L_t + \frac{\theta}{\theta-1}\frac{w_t}{Z_t^{1-\gamma} Z_t^{*\gamma}} Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t} = \frac{\theta}{\theta-1}(w_t L_t + w_t L_{*t})$,

which is in units of home country consumption.

Derivation of foreign GDP, y_t^*, i.e., output produced by home and foreign firms in the foreign country:

$y_t^* = RP_t^* Z_t^\gamma Z_t^{*1-\gamma} L_t^* + RP_{*t}^* Z_t^* L_{*t}^* = \frac{\theta}{\theta-1}\frac{w_t^*}{Z_t^\gamma Z_t^{*1-\gamma}} Z_t^\gamma Z_t^{*1-\gamma} L_t^* + \frac{\theta}{\theta-1}\frac{w_t^*}{Z_t^*} Z_t^* L_{*t}^* =$

$= \frac{\theta}{\theta-1}(w_t^* L_t^* + w_t^* L_{*t}^*)$,

which is in units of foreign country consumption.

Expression for $\frac{y_t}{y_t^*}$:

$\frac{y_t}{y_t^*} = \frac{RP_t Z_t L_t + RP_{*t} Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t}}{RP_t^* Z_t^\gamma Z_t^{*1-\gamma} L_t^* + RP_{*t}^* Z_t^* L_{*t}^*} = \frac{\frac{\theta}{\theta-1}(w_t L_t + w_t L_{*t})}{\frac{\theta}{\theta-1}(w_t^* L_t^* + w_t^* L_{*t}^*)} = \frac{w_t(L_t + L_{*t})}{w_t^*(L_t^* + L_{*t}^*)}$

Note that we should use the real exchange rate in the relative GDP expression but it cancels because it appears on both sides of the equation: $\frac{y_t}{Q_t y_t^*} = \frac{w_t(L_t + L_{*t})}{Q_t w_t^*(L_t^* + L_{*t}^*)}$

Next, expressions for w_t, w_t^*, $(L_t + L_{*t})$ and $(L_t^* + L_{*t}^*)$ are obtained. To get w_t, home labor supply and home labor demand are equated. Home labor supply was derived in Section 1.2 from home household FOCs as $L_t^s = (\frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi})^\varphi$. Home labor demand was derived above from firm FOCs in Section 1.3 as $L_t^d = RP_t^{-\omega}\frac{a(C_t+G_t)}{Z_t} + \frac{1-a}{a} RP_{*t}^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma} Z_t^{*\gamma}}$. Since $L_t^s = L_t^d$, i.e. labor-market clearing condition, it is possible to write:

$(\frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi})^\varphi = RP_t^{-\omega}\frac{a(C_t+G_t)}{Z_t} + \frac{1-a}{a} RP_{*t}^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma} Z_t^{*\gamma}}$

Substitute the expressions for RP_t and RP_{*t}:

$(\frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi})^\varphi = (\frac{\theta}{\theta-1}\frac{w_t}{Z_t})^{-\omega}\frac{a(C_t+G_t)}{Z_t} + \frac{1-a}{a}(\frac{\theta}{\theta-1}\frac{w_t}{Z_t^{1-\gamma} Z_t^{*\gamma}})^{-\omega}\frac{a(C_t+G_t)}{Z_t^{1-\gamma} Z_t^{*\gamma}}$

$w_t^{\omega+\varphi} = \chi^\varphi (\frac{\theta-1}{\theta})^\omega a(C_t+G_t) C_t^{\frac{\varphi}{\sigma}} [Z_t^{\omega-1} + \frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]$

$w_t = \chi^{\frac{\varphi}{\omega+\varphi}} (\frac{\theta-1}{\theta})^{\frac{\omega}{\omega+\varphi}} [a(C_t+G_t)]^{\frac{1}{\omega+\varphi}} C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}} [Z_t^{\omega-1} + \frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}$, which is the real wage in the home country in units of home consumption.

To get w_t^*, equate foreign labor supply and foreign labor demand. Labor supply can be derived from foreign household FOC as $L_t^* = (\frac{C_t^{*-\frac{1}{\sigma}} w_t^*}{\chi})^\varphi$ following the same steps as in Section 1.2. Labor demand was derived in Section 1.3 as $\frac{a}{1-a} RP_t^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^\gamma Z_t^{*1-\gamma}} + \frac{1-a}{1-a} RP_{*t}^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$.

$(\frac{C_t^{*-\frac{1}{\sigma}} w_t^*}{\chi})^\varphi = \frac{a}{1-a} RP_t^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^\gamma Z_t^{*1-\gamma}} + \frac{1-a}{1-a} RP_{*t}^{*-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$

$(\frac{C_t^{*-\frac{1}{\sigma}} w_t^*}{\chi})^{\varphi} = \frac{a}{1-a}(\frac{\theta}{\theta-1}\frac{w_t^*}{Z_t^{\gamma} Z_t^{*1-\gamma}})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^{\gamma} Z_t^{*1-\gamma}} + \frac{1-a}{1-a}(\frac{\theta}{\theta-1}\frac{w_t^*}{Z_t^*})^{-\omega}\frac{(1-a)(C_t^*+G_t^*)}{Z_t^*}$

$w_t^* = \chi^{\frac{\varphi}{\omega+\varphi}}(\frac{\theta-1}{\theta})^{\frac{\omega}{\omega+\varphi}}[(1-a)(C_t^* + G_t^*)]^{\frac{1}{\omega+\varphi}} C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1} + Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}$, which is real wage in the foreign country in units of foreign consumption.

$\frac{w_t}{w_t^*} = \frac{\chi^{\frac{\varphi}{\omega+\varphi}}(\frac{\theta-1}{\theta})^{\frac{\omega}{\omega+\varphi}}[a(C_t+G_t)]^{\frac{1}{\omega+\varphi}} C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}}[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}}{\chi^{\frac{\varphi}{\omega+\varphi}}(\frac{\theta-1}{\theta})^{\frac{\omega}{\omega+\varphi}}[(1-a)(C_t^*+G_t^*)]^{\frac{1}{\omega+\varphi}} C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}} =$
$\frac{[a(C_t+G_t)]^{\frac{1}{\omega+\varphi}} C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}}[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}}{[(1-a)(C_t^*+G_t^*)]^{\frac{1}{\omega+\varphi}} C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}}$

We can use this expression in the $\frac{y_t}{y_t^*}$ expression:

$\frac{y_t}{y_t^*} = \frac{w_t(L_t+L_{*t})}{w_t^*(L_t^*+L_{*t}^*)} = \frac{w_t(\frac{C_t^{-\frac{1}{\sigma}} w_t}{\chi})^{\varphi}}{w_t^*(\frac{C_t^{*-\frac{1}{\sigma}} w_t^*}{\chi})^{\varphi}} = \frac{w_t(C_t^{-\frac{1}{\sigma}} w_t)^{\varphi}}{w_t^*(C_t^{*-\frac{1}{\sigma}} w_t^*)^{\varphi}} = \frac{w_t^{1+\varphi} C_t^{-\frac{\varphi}{\sigma}}}{w_t^{*1+\varphi} C_t^{*-\frac{\varphi}{\sigma}}} =$
$= [\frac{C_t}{C_t^*}]^{\frac{\varphi(1-\omega)}{\sigma(\varphi+\omega)}}[\frac{a(C_t+G_t)}{(1-a)(C_t^*+G_t^*)}]^{\frac{1+\varphi}{\varphi+\omega}}[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}}{Z_t^{*\omega-1}+\frac{a}{1-a}(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}}]^{\frac{1+\varphi}{\varphi+\omega}}$

Note that the relative GDP can also be written as:

$\frac{y_t}{y_t^*} = [\frac{C_t}{C_t^*}]^{\frac{\varphi(1-\omega)}{\sigma(\varphi+\omega)}}(\frac{a}{1-a})^{\frac{1+\varphi}{\varphi+\omega}}[\frac{C_t+G_t}{C_t^*+G_t^*}]^{\frac{1+\varphi}{\varphi+\omega}}[\frac{\frac{1}{a}[aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]}{\frac{1}{1-a}[a(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}]}]^{\frac{1+\varphi}{\varphi+\omega}} =$
$= [\frac{C_t}{C_t^*}]^{\frac{\varphi(1-\omega)}{\sigma(\varphi+\omega)}}[\frac{C_t+G_t}{C_t^*+G_t^*}]^{\frac{1+\varphi}{\varphi+\omega}}[\frac{[aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]}{[a(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}]}]^{\frac{1+\varphi}{\varphi+\omega}}$

Parameters:

If $\varphi = 0$, i.e., inelastic labor:

$\frac{y_t}{y_t^*} = [\frac{a(C_t+G_t)}{(1-a)(C_t^*+G_t^*)}]^{\frac{1}{\omega}}[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}}{Z_t^{*\omega-1}+\frac{a}{1-a}(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}}]^{\frac{1}{\omega}}$

If also elasticity of substitution between home and foreign goods equals 1, i.e., $\omega = 1$:

$\frac{y_t}{y_t^*} = \frac{a}{1-a}\frac{C_t+G_t}{C_t^*+G_t^*}\frac{1-a}{a} = \frac{C_t+G_t}{C_t^*+G_t^*}$

If $\varphi > 0$, i.e., elastic labor, and $\omega = 1$:

$\frac{y_t}{y_t^*} = \frac{C_t+G_t}{C_t^*+G_t^*}$

We show that in this paper this result holds even if $\omega \neq 1$:

The expression $\frac{y_t}{y_t^*} = [\frac{C_t}{C_t^*}]^{\frac{\varphi(1-\omega)}{\sigma(\varphi+\omega)}}[\frac{C_t+G_t}{C_t^*+G_t^*}]^{\frac{1+\varphi}{\varphi+\omega}}[\frac{[aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}]}{[a(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}]}]^{\frac{1+\varphi}{\varphi+\omega}}$ can be combined with the expression derived below in Section 1.6 rewritten as: $(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}} = [\frac{C_t+G_t}{C_t^*+G_t^*}]^{-1}[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma} Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma} Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]^{\frac{1+\varphi}{\omega-1}}$

Then:

$$\frac{y_t}{y_t^*} = \Big[\frac{C_t+G_t}{C_t^*+G_t^*}\Big]^{-\frac{1-\omega}{\varphi+\omega}} \Big[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}\Big]^{\frac{1+\varphi}{\omega-1}\frac{1-\omega}{\varphi+\omega}} \Big[\frac{C_t+G_t}{C_t^*+G_t^*}\Big]^{\frac{1+\varphi}{\varphi+\omega}} \Big[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}\Big]^{\frac{1+\varphi}{\varphi+\omega}} =$$
$$= \Big[\frac{C_t+G_t}{C_t^*+G_t^*}\Big]^{\frac{(\omega-1)+(1+\varphi)}{\varphi+\omega}} + \Big[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}\Big]^{\frac{(1+\varphi)(1-\omega)+(1+\varphi)(\omega-1)}{(\omega-1)(\varphi+\omega)}} =$$
$$= \frac{C_t+G_t}{C_t^*+G_t^*}$$

1.6 More on real exchange rate, Q_t

From Section 1.1: $Q_t = \Big[\frac{a(\varepsilon_t P_{Ht}^*)^{1-\omega}+(1-a)(\varepsilon_t P_{Ft}^*)^{1-\omega}}{aP_{Ht}^{1-\omega}+(1-a)P_{Ft}^{1-\omega}}\Big]^{\frac{1}{1-\omega}}$.

$$Q_t^{1-\omega} = \frac{a(\varepsilon_t P_{Ht}^*)^{1-\omega}+(1-a)(\varepsilon_t P_{Ft}^*)^{1-\omega}}{aP_{Ht}^{1-\omega}+(1-a)P_{Ft}^{1-\omega}}$$

Use expressions for price indices:

$$Q_t^{1-\omega} = \frac{a(\varepsilon_t \frac{\theta}{\theta-1}\frac{W_t^*}{Z_t^{\gamma}Z_t^{*1-\gamma}})^{1-\omega}+(1-a)(\varepsilon_t \frac{\theta}{\theta-1}\frac{W_t^*}{Z_t^*})^{1-\omega}}{a(\frac{\theta}{\theta-1}\frac{W_t}{Z_t})^{1-\omega}+(1-a)(\frac{\theta}{\theta-1}\frac{W_t}{Z_t^{1-\gamma}Z_t^{*\gamma}})^{1-\omega}} = \Big(\frac{\varepsilon_t W_t^*}{W_t}\Big)^{1-\omega}\frac{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}} =$$
$$= \Big(\frac{\frac{\varepsilon_t W_t^*}{P_t^*}P_t^*}{\frac{W_t}{P_t}P_t}\Big)^{1-\omega}\frac{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}} = \Big(\frac{Q_t w_t^*}{w_t}\Big)^{1-\omega}\frac{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}} =$$
$$= \Big(\frac{Q_t w_t^*}{w_t}\Big)^{1-\omega}\frac{(1-a)[\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]}{a[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}]} = \Big(\frac{Q_t w_t^*}{w_t}\Big)^{1-\omega}\frac{1-a}{a}\frac{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}} =$$
$$= \Big(\frac{w_t}{Q_t w_t^*}\Big)^{\omega-1}\frac{1-a}{a}\frac{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}$$
$$\Big(\frac{w_t}{Q_t w_t^*}\Big)^{\omega-1} = \frac{a}{1-a}Q_t^{-(\omega-1)}\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}$$
$$\frac{w_t}{Q_t w_t^*} = \Big(\frac{a}{1-a}\Big)^{\frac{1}{\omega-1}}\frac{1}{Q_t}\Big[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}\Big]^{\frac{1}{\omega-1}}$$

Use the expression for $\frac{w_t}{w_t^*}$ from labor-clearing in Section 1.5 multiplied by $\frac{1}{Q_t}$:

$$\frac{w_t}{Q_t w_t^*} = \frac{1}{Q_t}\frac{[a(C_t+G_t)]^{\frac{1}{\omega+\varphi}}C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}}[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}}{[(1-a)(C_t^*+G_t^*)]^{\frac{1}{\omega+\varphi}}C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}}$$

Equate the two expressions:

$$\Big(\frac{a}{1-a}\Big)^{\frac{1}{\omega-1}}\frac{1}{Q_t}\Big[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}\Big]^{\frac{1}{\omega-1}} = \frac{1}{Q_t}\frac{[a(C_t+G_t)]^{\frac{1}{\omega+\varphi}}C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}}[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}}{[(1-a)(C_t^*+G_t^*)]^{\frac{1}{\omega+\varphi}}C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}}$$

$$\Big(\frac{a}{1-a}\Big)^{\frac{1}{\omega-1}}\Big[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}\Big]^{\frac{1}{\omega-1}} = \frac{[a(C_t+G_t)]^{\frac{1}{\omega+\varphi}}C_t^{\frac{\varphi}{\sigma(\omega+\varphi)}}[Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}]^{\frac{1}{\omega+\varphi}}}{[(1-a)(C_t^*+G_t^*)]^{\frac{1}{\omega+\varphi}}C_t^{*\frac{\varphi}{\sigma(\omega+\varphi)}}[\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}]^{\frac{1}{\omega+\varphi}}}$$

$$\Big(\frac{a}{1-a}\Big)^{\frac{1}{\omega-1}}\Big[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}\Big]^{\frac{1+\varphi}{(\omega-1)(\omega+\varphi)}} = \Big(\frac{a(C_t+G_t)}{(1-a)(C_t^*+G_t^*)}\Big)^{\frac{1}{\omega+\varphi}}\Big(\frac{C_t}{C_t^*}\Big)^{\frac{\varphi}{\sigma(\omega+\varphi)}}$$

This can be simplified further as follows but notice that the "Z" function changes (We denote it by Z'):

$$\Big(\frac{a}{1-a}\Big)^{\frac{\omega+\varphi}{\omega-1}}\Big[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}\Big]^{\frac{1+\varphi}{\omega-1}} = \frac{a(C_t+G_t)}{(1-a)(C_t^*+G_t^*)}\Big(\frac{C_t}{C_t^*}\Big)^{\frac{\varphi}{\sigma}}$$

$(\frac{a}{1-a})^{\frac{\omega+\varphi-(\omega-1)}{\omega-1}}[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}]^{\frac{1+\varphi}{\omega-1}} = \frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}}$
$(\frac{a}{1-a})^{\frac{\varphi+1}{\omega-1}}[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}]^{\frac{1+\varphi}{\omega-1}} = \frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}}$
$[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]^{\frac{1+\varphi}{\omega-1}} = \frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}}$

Parameters:

If $\varphi = 0$, i.e., inelastic labor:
$[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]^{\frac{1}{\omega-1}} = \frac{C_t+G_t}{C_t^*+G_t^*}$

If $\gamma = 1$:
$[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]$ reduces to 1.
$1 = \frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}}$
If also $\varphi = 0$: $C_t + G_t = C_t^* + G_t^*$

If $\gamma = 0$:
$[\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]$ reduces to $\frac{aZ_t^{\omega-1}+(1-a)Z_t^{\omega-1}}{aZ_t^{*\omega-1}+(1-a)Z_t^{*\omega-1}} = (\frac{Z_t}{Z_t^*})^{\omega-1}$.
$(\frac{Z_t}{Z_t^*})^{1+\varphi} = \frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}}$
If also $\varphi = 0$: $\frac{Z_t}{Z_t^*} = \frac{C_t+G_t}{C_t^*+G_t^*}$

1.7 Useful properties

It will be useful to take advantage of the fact that income distribution is determined by constant proportions, which is a feature of monopolistic competition models. Income consists of labor income and dividend income. As derived in Section 1.5, the home GDP, y_t, i.e., output produced by home and foreign firms in the home country, equals $\frac{\theta}{\theta-1}(w_tL_t + w_tL_{*t})$ in units of home country consumption. Therefore, the total home labor income equals $w_tL_t + w_tL_{*t} = \frac{\theta-1}{\theta}y_t$, which shows that the share of labor income in the home GDP is a constant proportion $\frac{\theta-1}{\theta}$. The profit of home firms, i.e., the profit generated by home firms in home and foreign countries, equals $d_t + d_t^* = y_t - y_t\frac{\theta-1}{\theta} = \frac{1}{\theta}y_t$, which shows that the share of firm profits, i.e., the dividend income, in the home GDP is a constant proportion $\frac{1}{\theta}$.

Similarly, foreign GDP y_t^*, i.e., output produced by home and foreign firms in the foreign country, equals $y_t^* = \frac{\theta}{\theta-1}(w_t^* L_t^* + w_t^* L_{*t}^*)$ in units of foreign country consumption. Labor income, therefore, equals $\frac{\theta-1}{\theta} y_t^*$. In units of home country consumption, this is $\frac{\theta-1}{\theta} y_t^* Q_t$. The profit of foreign firms, i.e., the profit generated by foreign firms in home and foreign countries, $d_{*t} + d_{*t}^*$, in units of home country consumption is then $\frac{1}{\theta} y_t^* Q_t$, which again shows that the share of firm profits, i.e., the dividend income, in the foreign GDP is a constant proportion $\frac{1}{\theta}$.

2 Model Solution

There are four variables that will determine the model solution: C_t^D, Q_t, y_t^D, and nfa_{t+1}.

2.1 Log-linearize Euler equations for consumption

Section 1.2 shows FOC wrt x_{t+1} combined with FOC wrt C_t, which gives the Euler equation:
$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} R_{t+1}\}$
Log-linearize this equation:
$-\frac{1}{\sigma}\widehat{C}_t = -\frac{1}{\sigma}E_t\widehat{C}_{t+1} + E_t\widehat{R}_{t+1}$

Similarly, the Euler equation for the foreign country is:
$C_t^{*-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{*-\frac{1}{\sigma}} R_{t+1}^f\} = \beta E_t\{C_{t+1}^{*-\frac{1}{\sigma}} R_{t+1}\frac{Q_t}{Q_{t-1}}\}$
where we used $R_t^f \equiv \frac{v_t^f + d_t^f + d_t^{f*}}{v_{t-1}^f}$ defined in Section 1.4 and $R_{t+1} = \frac{Q_{t+1}}{Q_t} R_{t+1}^f$ which gives $R_{t+1}^f = R_{t+1}\frac{Q_t}{Q_{t+1}}$
Log-linearize this equation:
$-\frac{1}{\sigma}\widehat{C}_t^* = -\frac{1}{\sigma}E_t\widehat{C}_{t+1}^* + E_t\widehat{R}_{t+1} + E_t\widehat{Q}_t - E_t\widehat{Q}_{t+1}$

Subtract the home and foreign equations:
$-\frac{1}{\sigma}(\widehat{C}_t - \widehat{C}_t^*) = -\frac{1}{\sigma}E_t(\widehat{C}_{t+1} - \widehat{C}_{t+1}^*) + E_t(\widehat{R}_{t+1} - \widehat{R}_{t+1} - (\widehat{Q}_t - \widehat{Q}_{t+1}))$
$-\frac{1}{\sigma}(\widehat{C}_t - \widehat{C}_t^*) = -\frac{1}{\sigma}E_t(\widehat{C}_{t+1} - \widehat{C}_{t+1}^*) + E_t(\widehat{Q}_{t+1} - \widehat{Q}_t)$
$-\frac{1}{\sigma}(\widehat{C}_t^D) = -\frac{1}{\sigma}E_t(\widehat{C}_{t+1}^D) + E_t(\widehat{Q}_{t+1} - \widehat{Q}_t)$
$\widehat{C}_t^D = E_t(\widehat{C}_{t+1}^D) - \sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t)$

$E_t(\widehat{C}_{t+1}^D - \widehat{C}_t^D) = \sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t)$

2.2 Log-linearize expression from Section 1.6 and find elasticities of $\widehat{C}_t^D$

This derivation finds elasticities of $\widehat{C}_t^D$:

$\frac{C_t+G_t}{C_t^*+G_t^*}(\frac{C_t}{C_t^*})^{\frac{\varphi}{\sigma}} = [\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]^{\frac{1+\varphi}{\omega-1}}$

$\log(C_t+G_t)-\log(C_t^*+G_t^*)+\frac{\varphi}{\sigma}(\log C_t-\log C_t^*) = \frac{1+\varphi}{\omega-1}[\log(aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1})-\log(a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1})]$

$\frac{dC_t+dG_t}{C+G} - \frac{dC_t^*+dG_t^*}{C+G} + \frac{\varphi}{\sigma}(\frac{dC_t}{C} - \frac{dC_t^*}{C}) = \frac{1+\varphi}{\omega-1}[a(\omega-1)dZ_t + (1-a)(\omega-1)((1-\gamma)dZ_t + \gamma dZ_t^*) - [a(\omega-1)(\gamma dZ_t + (1-\gamma)dZ_t^*) +$
$(1-a)(\omega-1)dZ_t^*]]$

Use $Z = Z^*$, which is true in the symmetric steady state. Normalize $Z = Z^*$ to 1.

$\frac{dC_t\frac{C}{C}+dG_t\frac{G}{G}}{C+G} - \frac{dC_t^*\frac{C}{C}+dG_t^*\frac{G}{G}}{C+G} + \frac{\varphi}{\sigma}\widehat{C}_t^D = \frac{1+\varphi}{\omega-1}[a(\omega-1)\widehat{Z}_t + (1-a)(\omega-1)((1-\gamma)\widehat{Z}_t + \gamma\widehat{Z}_t^*) - [a(\omega-1)(\gamma\widehat{Z}_t + (1-\gamma)\widehat{Z}_t^*) + (1-$
$a)(\omega-1)\widehat{Z_t^*}]$

$\frac{C}{C+G}(\widehat{C}_t - \widehat{C}_t^*) + \frac{G}{C+G}(\widehat{G}_t - \widehat{G}_t^*) + \frac{\varphi}{\sigma}\widehat{C}_t^D = \frac{1+\varphi}{\omega-1}[a(\omega-1)\widehat{Z}_t + (1-a)(\omega-1)(1-\gamma)\widehat{Z}_t + (1-a)(\omega-1)\gamma\widehat{Z}_t^* - a(\omega-1)\gamma\widehat{Z}_t -$
$a(\omega-1)(1-\gamma)\widehat{Z}_t^* - (1-a)(\omega-1)\widehat{Z}_t^*]$

Use $y = C + G$. Since $y = 1$, $C + G = 1$ and $C = 1 - G$. Then,

$(1-G)(\widehat{C}_t-\widehat{C}_t^*)+G(\widehat{G}_t-\widehat{G}_t^*)+\frac{\varphi}{\sigma}\widehat{C}_t^D = \frac{1+\varphi}{\omega-1}[a(\omega-1)(1-\gamma)\widehat{Z}_t+(1-a)(\omega-1)(1-\gamma)\widehat{Z}_t-(1-a)(\omega-1)(1-\gamma)\widehat{Z}_t^*-a(\omega-1)(1-\gamma)\widehat{Z}_t^*$

$(1-G)\widehat{C}_t^D + G\widehat{G}_t^D + \frac{\varphi}{\sigma}\widehat{C}_t^D = \frac{1+\varphi}{\omega-1}(\omega-1)(1-\gamma)(\widehat{Z}_t - \widehat{Z}_t^*)$

$(1-G)\widehat{C}_t^D + G\widehat{G}_t^D + \frac{\varphi}{\sigma}\widehat{C}_t^D = (1+\varphi)(1-\gamma)\widehat{Z}_t^D$

$(1-G+\frac{\varphi}{\sigma})\widehat{C}_t^D + = (1+\varphi)(1-\gamma)\widehat{Z}_t^D - G\widehat{G}_t^D$

$\widehat{C}_t^D = \frac{(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D - \frac{G}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D$

$\widehat{C}_t^D = \eta_{C^D Z^D}\widehat{Z}_t^D + \eta_{C^D G^D}\widehat{G}_t^D$

If $G = 0$ (i.e., no fiscal shocks), $\widehat{C}_t^D = \frac{(1+\varphi)(1-\gamma)}{1+\frac{\varphi}{\sigma}}\widehat{Z}_t^D$

If $G = 0$ and $\varphi = 0$ (i.e., inelastic labor) $\widehat{C}_t^D = (1-\gamma)\widehat{Z}_t^D$

If $G = 0$, $\varphi = 0$, and $\gamma = 1$, $\widehat{C}_t^D = 0$.

If $G = 0$, $\varphi = 0$, and $\gamma = 0$, $\widehat{C}_t^D = \widehat{Z}_t^D$.

If $G = 0$ and $\gamma = 1$, $\widehat{C}_t^D = 0$ regardless of φ.

If $G \neq 0$ and $\varphi = 0$, $\widehat{C}_t^D = \frac{(1-\gamma)}{1-G}\widehat{Z}_t^D - \frac{G}{1-G}\widehat{G}_t^D$

If $G \neq 0$, $\varphi = 0$ and $\gamma = 1$, $\widehat{C}_t^D = -\frac{G}{1-G}\widehat{G}_t^D$. If $\gamma = 0$, $\widehat{C}_t^D = \frac{1}{1-G}\widehat{Z}_t^D - \frac{G}{1-G}\widehat{G}_t^D$.

2.3 Find elasticities of $\widehat{Q}_t$

This derivation uses the log-linearized Euler equations from Section 2.1 and $\widehat{C}_t^D$ from Section 2.2 to find elasticities of $\widehat{Q}_t$:

$E_t(\widehat{C}_{t+1}^D - \widehat{C}_t^D) = \sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t)$ from Section 2.1.

Combine with $\widehat{C}_t^D = \eta_{C^D Z^D}\widehat{Z}_t^D + \eta_{C^D G^D}\widehat{G}_t^D$ from 2.2.

$\sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t) = E_t[\eta_{C^D Z^D}(\widehat{Z}_{t+1}^D - \widehat{Z}_t^D) + \eta_{C^D G^D}(\widehat{G}_{t+1}^D - \widehat{G}_t^D)]$.

$\widehat{Z}_{t+1}^D = \phi_Z \widehat{Z}_t^D + \widehat{\xi}_{Z^D t+1}$ and $\widehat{G}_{t+1}^D = \phi_G \widehat{G}_t^D + \widehat{\xi}_{G^D t+1}$ where ϕ_Z and ϕ_G denote the persistence of relative productivity and government spending shocks defined as the percentage deviations from the steady state, so:

$\sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t) = \eta_{C^D Z^D}(\phi_Z - 1)\widehat{Z}_t^D + \eta_{C^D G^D}(\phi_G - 1)\widehat{G}_t^D$

$\sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t) = -\eta_{C^D Z^D}(1 - \phi_Z)\widehat{Z}_t^D - \eta_{C^D G^D}(1 - \phi_G)\widehat{G}_t^D$

$\widehat{Q}_t = \eta_{QZ^D}\widehat{Z}_t^D + \eta_{QG^D}\widehat{G}_t^D$

$\sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t) = \sigma(\phi_Z - 1)\eta_{QZ^D}\widehat{Z}_t^D + \sigma(\phi_G - 1)\eta_{QG^D}\widehat{G}_t^D$

$\sigma E_t(\widehat{Q}_{t+1} - \widehat{Q}_t) = -\sigma(1 - \phi_Z)\eta_{QZ^D}\widehat{Z}_t^D - \sigma(1 - \phi_G)\eta_{QG^D}\widehat{G}_t^D$

$-\eta_{C^D Z^D}(1 - \phi_Z)\widehat{Z}_t^D - \eta_{C^D G^D}(1 - \phi_G)\widehat{G}_t^D = -\sigma\eta_{QZ^D}(1 - \phi_Z)\widehat{Z}_t^D - \sigma\eta_{QG^D}(1 - \phi_G)\widehat{G}_t^D$

$\eta_{QZ^D} = \frac{1}{\sigma}\eta_{C^D Z^D}$

$\eta_{QG^D} = \frac{1}{\sigma}\eta_{C^D G^D}$

Notice:

$\widehat{Q}_t = \eta_{QZ^D}\widehat{Z}_t^D + \eta_{QG^D}\widehat{G}_t^D$

Combine with $\widehat{C}_t^D = \eta_{C^D Z^D}\widehat{Z}_t^D + \eta_{C^D G^D}\widehat{G}_t^D$ from Section 2.2.

Because $\eta_{QZ^D} = \frac{1}{\sigma}\eta_{C^D Z^D}$ and $\eta_{QG^D} = \frac{1}{\sigma}\eta_{C^D G^D}$,

$\widehat{Q}_t = \frac{1}{\sigma}\widehat{C}_t^D$

$\widehat{C}_t^D = \sigma\widehat{Q}_t$, which shows complete markets risk-sharing. In other words, what gives the risk-sharing is the movement in the real exchange rate.

$\widehat{Q}_t = \frac{1}{\sigma}\widehat{C}_t^D$

2.4 Log-linearize relative GDP from Section 1.5 and find elasticities of $\widehat{y}_t^D$

This derivation log-linearizes $\frac{y_t}{y_t^*}$ and then uses $\widehat{C}_t^D$ to find elasticities of $\widehat{y}_t^D$

First, log-linearize $\frac{y_t}{y_t^*}$. The relative GDP was derived in Section 1.5 as $\frac{y_t}{y_t^*} = \frac{C_t+G_t}{C_t^*+G_t^*}$. We log-linearize following the derivation in Section 2.2:

$\frac{C_t+G_t}{C_t^*+G_t^*}$ becomes $\log(C_t+G_t) - \log(C_t^*+G_t^*)$ and then $\frac{dC_t+dG_t}{C+G} - \frac{dC_t^*+dG_t^*}{C+G} = \frac{dC_t\frac{C}{C}+dG_t\frac{G}{G}}{C+G} - \frac{dC_t^*\frac{C}{C}+dG_t^*\frac{G}{G}}{C+G} = \frac{C}{C+G}(\widehat{C}_t - \widehat{C}_t^*) + \frac{G}{C+G}(\widehat{G}_t - \widehat{G}_t^*) = (1-G)(\widehat{C}_t - \widehat{C}_t^*) + G(\widehat{G}_t - \widehat{G}_t^*) =$

$= (1-G)\widehat{C}_t^D + G\widehat{G}_t^D$

Then, find elasticities: $\widehat{y}_t^D = (1-G)\widehat{C}_t^D + G\widehat{G}_t^D$

Use $\widehat{C}_t^D = \frac{(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D - \frac{G}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D$ from Section 2.2:

$\widehat{y}_t^D = (1-G)[\frac{(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D - \frac{G}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D] + G\widehat{G}_t^D =$

$= \frac{(1-G)(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D + \frac{G(1-G+\frac{\varphi}{\sigma})-G(1-G)}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D =$

$= \frac{(1-G)(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D + \frac{G\frac{\varphi}{\sigma}}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D =$

$= \eta_{y^D Z^D}\widehat{Z}_t^D + \eta_{y^D G^D}\widehat{G}_t^D$

If $G=0$, $\widehat{y}_t^D = \frac{(1+\varphi)(1-\gamma)}{1+\frac{\varphi}{\sigma}}\widehat{Z}_t^D$

If $G=0$ and $\gamma=1$, $\widehat{y}_t^D = 0$

If $G=0$ and $\gamma=0$, $\widehat{y}_t^D = \frac{1+\varphi}{1+\frac{\varphi}{\sigma}}\widehat{Z}_t^D$

If $G=0$ and $\varphi=0$, $\widehat{y}_t^D = (1-\gamma)\widehat{Z}_t^D$

If $G\neq 0$ and $\varphi=0$, $\widehat{y}_t^D = (1-\gamma)\widehat{Z}_t^D$

If $\gamma=0$, $\widehat{y}_t^D = \frac{(1-G)(1+\varphi)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}_t^D + \frac{G\frac{\varphi}{\sigma}}{1-G+\frac{\varphi}{\sigma}}\widehat{G}_t^D$

If $\gamma=0$ and $\varphi=0$, $\widehat{y}_t^D = \widehat{Z}_t^D$

2.5 Log-linearize the wage differential and labor differential

This section shows log-linearized wage differential and labor differential. These differentials are used in the impulse response functions in Section 4 of the paper.

The wage differential is derived in Section 1.6 as:

$\frac{w_t}{w_t^*} = (\frac{a}{1-a})^{\frac{1}{\omega-1}}[\frac{Z_t^{\omega-1}+\frac{1-a}{a}(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{\frac{a}{1-a}(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+Z_t^{*\omega-1}}]^{\frac{1}{\omega-1}} = [\frac{aZ_t^{\omega-1}+(1-a)(Z_t^{1-\gamma}Z_t^{*\gamma})^{\omega-1}}{a(Z_t^{\gamma}Z_t^{*1-\gamma})^{\omega-1}+(1-a)Z_t^{*\omega-1}}]^{\frac{1}{\omega-1}}$

This can be log-linearized using the same steps as in Section 2.2:

$\widehat{w}_t - \widehat{w^*}_t = (1-\gamma)(\widehat{Z}_t - \widehat{Z}_t^*)$, which is $\widehat{w}_t^D = (1-\gamma)\widehat{Z}_t^D$.

The labor differential follows from the useful properties in Section 1.7. The home GDP, y_t, is $\frac{\theta}{\theta-1}(w_tL_t + w_tL_{*t})$ in units of home country consumption. The foreign GDP y_t^*, is $y_t^* = \frac{\theta}{\theta-1}(w_t^*L_t^* + w_t^*L_{*t}^*)$ in units of foreign country consumption. Therefore, $\frac{w_tL_t^{total}}{w_t^*L_t^{*total}} = \frac{y_t}{y_t^*}$.

Log-linearized: $\widehat{L}_t^{total,D} = \widehat{y}_t^D - \widehat{w}_t^D$.

2.6 Log-linearize NFA LOM

This derivation uses the NFA LOM from Section 1.4 to find the solution for $n\widehat{f}a_{t+1}$:

$nfa_{t+1} = R_t^D \alpha_t + R_t nfa_t + (1-a)[(y_t - Q_t y_t^{*f}) - (C_t - Q_t C_t^{*f}) - (G_t - Q_t G_t^{*f})]$

$dnfa_{t+1} = dR_t^D \alpha + R^D d\alpha_t + dR_t nfa + R dnfa_t + (1-a)[dy_t - (dQ_t y^{*f} + Q dy_t^{*f}) - (dC_t - (dQ_t C^{*f} + Q dC_t^{*f})) - (dG_t - (dQ_t G^{*f} + Q dG_t^{*f})]$

Use $R^D = 0$ and $nfa = 0$:

$dnfa_{t+1} = dR_t^D \alpha + R dnfa_t + (1-a)[dy_t - (dQ_t y^{*f} + Q dy_t^{*f}) - (dC_t - (dQ_t C^{*f} + Q dC_t^{*f})) - (dG_t - (dQ_t G^{*f} + Q dG_t^{*f})]$

Use $Q = 1$ because it holds in the symmetric steady state, and net foreign assets equal 0:

$dnfa_{t+1} = dR_t^D \alpha + R dnfa_t + (1-a)[(dy_t - (dQ_t y^{*f} + dy_t^{*f}) - (dC_t - (dQ_t C^{*f} + dC_t^{*f})) - (dG_t - (dQ_t G^{*f} + dG_t^{*f})]$

Notice that we are subtracting dy_t and dy_t^{*f} that are in units of home consumption and foreign consumption, respectively. We can subtract these terms because we already accounted for the different units by including the real exchange rate. This works because in the symmetric steady state, the real exchange rate terms drop out ($Q = 1$). This is used later on in other derivations, for example, the derivation of the differential in equity values, $\widehat{v}_t^D$.

$dnfa_{t+1} = dR_t^D \alpha + R dnfa_t + (1-a)[(dy_t^D - dQ_t y^{*f}) - (dC_t^D - dQ_t C^{*f}) - (dG_t^D - dQ_t G^{*f})]$

Divide by C. Use $C = 1 - G$, which comes from $y = C + G$ combined with $y = 1$:

$\frac{dnfa_{t+1}}{C} = \frac{dR_t^D \alpha}{1-G} + \frac{R dnfa_t}{C} + (1-a)[(\frac{dy_t^D}{1-G} - \frac{dQ_t y^{*f}}{1-G}) - (\frac{dC_t^D}{C} - \frac{dQ_t C^{*f}}{C}) - (\frac{dG_t^D}{1-G} - \frac{dQ_t G^{*f}}{1-G})]$

$n\widehat{f}a_{t+1} = \frac{dR_t^D \alpha}{1-G}\frac{R}{R} + \frac{\frac{1}{\beta} dnfa_t}{C} + (1-a)[(\frac{dy_t^D}{1-G}\frac{y}{y} - \frac{dQ_t y^{*f}}{1-G}) - (\widehat{C}_t^D - \frac{dQ_t C^{*f}}{C}) - (\frac{dG_t^D}{1-G}\frac{G}{G} - \frac{dQ_t G^{*f}}{1-G})]$

$n\widehat{f}a_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}_t^D + \frac{1}{\beta} n\widehat{f}a_t + \frac{1-a}{1-G}\widehat{y}_t^D - (1-a)\widehat{C}_t^D - \frac{(1-a)G}{1-G}\widehat{G}_t^D + (1-a)[-\frac{dQ_t y^{*f}}{1-G} + \frac{dQ_t C^{*f}}{C} + \frac{dQ_t G^{*f}}{1-G}]$

$n\widehat{f}a_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}_t^D + \frac{1}{\beta} n\widehat{f}a_t + \frac{1-a}{1-G}\widehat{y}_t^D - (1-a)\widehat{C}_t^D - \frac{(1-a)G}{1-G}\widehat{G}_t^D + (1-a)[-\frac{dQ_t y^{*f}}{1-G}\frac{Q}{Q} + \frac{dQ_t C^{*f}}{C}\frac{Q}{Q} + \frac{dQ_t G^{*f}}{1-G}\frac{Q}{Q}]$

Use $Q = 1$:

$n\widehat{f}a_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}_t^D + \frac{1}{\beta} n\widehat{f}a_t + \frac{1-a}{1-G}\widehat{y}_t^D - (1-a)\widehat{C}_t^D - \frac{(1-a)G}{1-G}\widehat{G}_t^D + (1-a)[-\frac{\widehat{Q}_t y^{*f}}{1-G} + \frac{\widehat{Q}_t C^{*f}}{C} + \frac{\widehat{Q}_t G^{*f}}{1-G}]$

Use $y^{*f} = 1$:

$n\widehat{f}a_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}_t^D + \frac{1}{\beta} n\widehat{f}a_t + \frac{1-a}{1-G}\widehat{y}_t^D - (1-a)\widehat{C}_t^D - \frac{(1-a)G}{1-G}\widehat{G}_t^D + (1-a)[-\frac{\widehat{Q}_t}{1-G} + \frac{\widehat{Q}_t C^{*f}}{C} + \frac{\widehat{Q}_t G^{*f}}{1-G}]$

Use $C = 1 - G$. Use $y^{*f} = C^{*f} + G^{*f}$ combined with $y^{*f} = 1$:

$\widehat{nfa}_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}^D_t + \frac{1}{\beta}\widehat{nfa}_t + \frac{1-a}{1-G}\widehat{y}^D_t - (1-a)\widehat{C}^D_t - \frac{(1-a)G}{1-G}\widehat{G}^D_t + (1-a)[-\frac{\widehat{Q}_t}{1-G} + \frac{\widehat{Q}_t(1-G^{*f})}{1-G} + \frac{\widehat{Q}_t G^{*f}}{1-G}]$
$\widehat{nfa}_{t+1} = \frac{\alpha}{\beta(1-G)}\widehat{R}^D_t + \frac{1}{\beta}\widehat{nfa}_t + \frac{1-a}{1-G}\widehat{y}^D_t - (1-a)\widehat{C}^D_t - \frac{(1-a)G}{1-G}\widehat{G}^D_t$
Define excess return shock $\widehat{\xi}_t \equiv \frac{\alpha}{\beta(1-G)}\widehat{R}^D_t$ as in GLR.
$\widehat{nfa}_{t+1} = \frac{1}{\beta}\widehat{nfa}_t + \widehat{\xi}_t + \frac{1-a}{1-G}\widehat{y}^D_t - (1-a)\widehat{C}^D_t - \frac{(1-a)G}{1-G}\widehat{G}^D_t$
Note that this expression is identical to GLR.
Given solutions for $\widehat{C}^D_t$ and $\widehat{y}^D_t$ (that are functions of $\widehat{Z}^D_t$ and $\widehat{G}^D_t$), this gives solution for $\widehat{nfa}_{t+1}$ as a function of $\widehat{nfa}_t$, $\widehat{\xi}_t$, $\widehat{Z}^D_t$, and $\widehat{G}^D_t$.
Notice there is no real exchange rate in the log-linearized LOM for NFA. This is because of the symmetry of the steady state.

We can use $\widehat{y}^D_t$ derived in Section 2.4 as $\widehat{y}^D_t = (1-G)\widehat{C}^D_t + G\widehat{G}^D_t$ to substitute here:
$\widehat{nfa}_{t+1} = \frac{1}{\beta}\widehat{nfa}_t + \frac{\alpha}{\beta(1-G)}\widehat{R}^D_t + \frac{1-a}{1-G}[(1-G)\widehat{C}^D_t + G\widehat{G}^D_t] - (1-a)\widehat{C}^D_t - \frac{(1-a)G}{1-G}\widehat{G}^D_t$
$\widehat{nfa}_{t+1} = \frac{1}{\beta}\widehat{nfa}_t + \frac{\alpha}{\beta(1-G)}\widehat{R}^D_t$
$\alpha = 0$ is required for $nfa_{t+1} = nfa_t \ \forall t$.

2.7 Find elasticities of $\widehat{v}^D_t$

Euler equations from Section 1.2:
$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}\frac{v_{t+1}+d_{t+1}+d^*_{t+1}}{v_t}\} \equiv \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}R_{t+1}\}$
$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}\frac{v^*_{t+1}+d_{*t+1}+d^*_{*t+1}}{v^*_t}\} \equiv \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}R^*_{t+1}\}$
Both of these equations are in units of home country consumption.

Excess return from holding foreign equity is defined in Section 1.4 as $R^D_t = R^*_t - R_t$.
Use $E_t\widehat{R}^D_{t+1} = 0$.

Log-linearize Euler equations for home firm and foreign firm shares from Section 1.2.
Starting with home firm shares:
$C_t^{-\frac{1}{\sigma}}v_t = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}(v_{t+1} + d_{t+1} + d^*_{t+1})\}$
$dv_t = \beta E_t dv_{t+1} + \beta E_t dd_{t+1} + \beta E_t dd^*_{t+1}$
Divide by v:

$\frac{dv_t}{v} = \frac{\beta d E_t v_{t+1}}{v} + \frac{\beta d E_t d_{t+1}}{v} + \frac{\beta d E_t d^*_{t+1}}{v}$

$\widehat{v}_t = \beta E_t \widehat{v}_{t+1} + \beta E_t \widehat{d}_{t+1} \frac{d}{v} + \beta E_t \widehat{d}^*_{t+1} \frac{d^*}{v}$

From Section 1.7, the following holds: $d_t + d^*_t = \frac{1}{\theta} y_t$. Due to the assumption $y_t = 1$, it is possible to write: $d_t + d^*_t = \frac{1}{\theta}$. In steady state, the Euler equation for home shares becomes $v = \beta v + \beta d + \beta d^*$, which becomes $v = \beta v + \beta \frac{1}{\theta}$ which can be written as $v(1-\beta) = \frac{\beta}{\theta}$ which can be written as $v = \frac{\beta}{\theta(1-\beta)}$.

$\widehat{v}_t = \beta E_t \widehat{v}_{t+1} + \beta E_t \widehat{d}_{t+1} \frac{d}{\frac{\beta}{1-\beta}\frac{1}{\theta}} + \beta E_t \widehat{d}^*_{t+1} \frac{d^*}{\frac{\beta}{1-\beta}\frac{1}{\theta}} = \beta E_t \widehat{v}_{t+1} + (1-\beta) E_t \widehat{d}_{t+1} d\theta + (1-\beta) E_t \widehat{d}^*_{t+1} d^* \theta = \beta E_t \widehat{v}_{t+1} + (1-\beta)\theta E_t \widehat{d}_{t+1} d + (1-\beta)\theta E_t \widehat{d}^*_{t+1}(1-d)$, which does not lead to a convenient format as in GLR. Therefore, we define total home firm profit as $\overline{d}_{t+1} = d_{t+1} + d^*_{t+1}$.

We can then write the Euler equation for home firm shares as: $C_t^{-\frac{1}{\sigma}} v_t = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} R_{t+1}\} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}} (v_{t+1} + \overline{d}_{t+1})\}$

$dv_t = \beta E_t dv_{t+1} + \beta E_t d\overline{d}_{t+1}$

Divide by v:

$\frac{dv_t}{v} = \frac{\beta d E_t v_{t+1}}{v} + \frac{\beta d E_t \overline{d}_{t+1}}{v}$

$\widehat{v}_t = \beta E_t \widehat{v}_{t+1} + \beta E_t \widehat{d}_{t+1} \frac{\overline{d}}{v}$

From Section 1.7, the following holds: $d_t + d^*_t \equiv \overline{d}_t = \frac{1}{\theta} y_t$. Again, due to the assumption $y_t = 1$, it is possible to write: $\overline{d}_t = \frac{1}{\theta}$, which in steady state becomes $\overline{d} = \frac{1}{\theta}$. In steady state, the Euler equation becomes $v = \beta v + \beta \overline{d}$, which becomes $v = \beta v + \beta \frac{1}{\theta}$ which can be written as $v(1-\beta) = \frac{\beta}{\theta}$ which can be written as $v = \frac{\beta}{\theta(1-\beta)}$.

$\widehat{v}_t = \beta E_t \widehat{v}_{t+1} + \beta E_t \widehat{\overline{d}}_{t+1} \frac{\frac{1}{\theta}}{\frac{\beta}{(1-\beta)\theta}}$

$\widehat{v}_t = \beta E_t \widehat{v}_{t+1} + E_t \widehat{\overline{d}}_{t+1}(1-\beta)$

Following the same steps for the foreign firm's shares:

$\widehat{v}^*_t = \beta E_t \widehat{v}^*_{t+1} + E_t \widehat{\overline{d}}^*_{t+1}(1-\beta)$ where $\overline{d}^*_t$ is defined as $d_{*t} + d^*_{*t}$, i.e., total profit of the foreign firm.

Subtracting expressions for the home firm and foreign firm shares:

$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)\widehat{\overline{d}}^D_{t+1}]$ where the log-linearized difference between total profits generated by the home firm and total profits generated by the foreign firm, $\widehat{\overline{d}}^D_{t+1}$, is defined as $\widehat{\overline{d}}_{t+1} - \widehat{\overline{d}}^*_{t+1} = (\widehat{d_{t+1} + d^*_{t+1}}) - (\widehat{d_{*t+1} + d^*_{*t+1}})$. Note that this is similar to Equation (43) on p. A-4 of GLR.

Next, we obtain an expression for $\widehat{\overline{d}}^D_{t+1}$. Here, we take advantage of the useful properties from Section 1.7. Since $\overline{d}_t = d_t + d^*_t = \frac{1}{\theta} y_t$ and $\overline{d}^*_t = d^*_t + d^*_{*t} = \frac{1}{\theta} y_{*t} Q_t$ in units of home country consumption, it is possible to write $\frac{\overline{d}_t}{\overline{d}^*_t} = \frac{d_t + d^*_t}{d_{*t} + d^*_{*t}} = \frac{\frac{1}{\theta} y_t}{\frac{1}{\theta} y^*_t Q_t}$, which means $\frac{\overline{d}_t}{\overline{d}^*_t} = \frac{y_t}{y^*_t Q_t}$.
Roll it forward by one period: $\frac{\overline{d}_{t+1}}{\overline{d}^*_{t+1}} = \frac{y_{t+1}}{y^*_{t+1} Q_{t+1}}$.
Log-linearizing gives $\widehat{\overline{d}}^D_{t+1} = \widehat{y}_{t+1} - (\widehat{y}^*_{t+1} + \widehat{Q}_{t+1})$.

Substitute into $\widehat{v}^D_t$:
$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)(\widehat{y}_{t+1} - (\widehat{y}^*_{t+1} + \widehat{Q}_{t+1}))]$
Notice: This combines $E_t \widehat{R}^D_{t+1} = 0$ and $\widehat{R}^D_t = -[\beta \widehat{v}^D_t + (1-\beta)(\widehat{y}_t - (\widehat{y}^*_t + \widehat{Q_t}))] + \widehat{v}^D_{t-1} =$
$= -[\beta \widehat{v}^D_t + (1-\beta)(\widehat{y}^D_t - \widehat{Q_t})] + \widehat{v}^D_{t-1}$
$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)(\widehat{y}^D_{t+1} - \widehat{Q}_{t+1})]$
$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)(\eta_{y^D Z^D} \widehat{Z}^D_{t+1} + \eta_{y^D G^D} \widehat{G}^D_{t+1} - \eta_{Q Z^D} \widehat{Z}^D_{t+1} - \eta_{Q G^D} \widehat{G}^D_{t+1}]$
$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)(\eta_{y^D Z^D} \widehat{Z}^D_{t+1} + \eta_{y^D G^D} \widehat{G}^D_{t+1} - \frac{1}{\sigma}\eta_{C^D Z^D} \widehat{Z}^D_{t+1} - \frac{1}{\sigma}\eta_{C^D G^D} \widehat{G}^D_{t+1}]$
$\widehat{v}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)((\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D}) \widehat{Z}^D_{t+1} + (\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D}) \widehat{G}^D_{t+1}]$

$\widehat{v}^D_t = \eta_{v^D Z^D} \widehat{Z}^D_t + \eta_{v^D G^D} \widehat{G}^D_t$
$\eta_{v^D Z^D} \widehat{Z}^D_t + \eta_{v^D G^D} \widehat{G}^D_t = E_t[\beta \widehat{v}^D_{t+1} + (1-\beta)((\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D}) \widehat{Z}^D_{t+1} + (\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D}) \widehat{G}^D_{t+1}]$
$\eta_{v^D Z^D} \widehat{Z}^D_t + \eta_{v^D G^D} \widehat{G}^D_t = \beta(\eta_{v^D Z^D} \phi_Z \widehat{Z}^D_t + \eta_{v^D G^D} \phi_G \widehat{G}^D_t) + (1-\beta)[(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D}) \phi_Z \widehat{Z}^D_t +$
$(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D}) \phi_G \widehat{G}^D_t]$
where we used $\widehat{Z}^D_{t+1} = \phi_Z \widehat{Z}^D_t + \widehat{\xi}_{Z^D t+1}$ and $\widehat{G}^D_{t+1} = \phi_G \widehat{G}^D_t + \widehat{\xi}_{G^D t+1}$
Match the coefficients:
$\eta_{v^D Z^D} = \beta \eta_{v^D Z^D} \phi_Z + (1-\beta)(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D}) \phi_Z$
$(1-\beta\phi_Z)\eta_{v^D Z^D} = (1-\beta)(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D}) \phi_Z$
$\eta_{v^D Z^D} = \frac{(1-\beta)\phi_Z(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})}{1-\beta\phi_Z}$
$\eta_{v^D G^D} = \beta \eta_{v^D G^D} \phi_G + (1-\beta)(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D}) \phi_G$
$(1-\beta\phi_G)\eta_{v^D G^D} = (1-\beta)(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D}) \phi_G$
$\eta_{v^D G^D} = \frac{(1-\beta)\phi_G(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})}{1-\beta\phi_G}$

2.8 Show that excess return $\widehat{R}_t^D$ is a linear function of innovations to relative productivity and government spending

From Section 2.7:

$\widehat{R}^D_{t+1} = -[\beta\widehat{v}^D_{t+1} + (1-\beta)(\widehat{y}^D_{t+1} - \widehat{Q}_{t+1})] + \widehat{v}^D_t =$

$= -[\beta(\eta_{v^D Z^D}\widehat{Z}^D_{t+1} + \eta_{v^D G^D}\widehat{G}^D_{t+1}) + (1-\beta)(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})\widehat{Z}^D_{t+1} + (1-\beta)(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})\widehat{G}^D_{t+1}] +$
$\eta_{v^D Z^D}\widehat{Z}^D_t + \eta_{v^D G^D}\widehat{G}^D_t =$

$= -[\beta\frac{(1-\beta)\phi_Z(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})}{1-\beta\phi_Z}\widehat{Z}^D_{t+1} + \beta\frac{(1-\beta)\phi_G(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})}{1-\beta\phi_G}\widehat{G}^D_{t+1} + (1-\beta)(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})\widehat{Z}^D_{t+1} +$
$(1-\beta)(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})\widehat{G}^D_{t+1}] + \frac{(1-\beta)\phi_Z(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})}{1-\beta\phi_Z}\widehat{Z}^D_t + \frac{(1-\beta)\phi_G(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})}{1-\beta\phi_G}\widehat{G}^D_t =$

$= -[\beta\eta_{v^D Z^D}\widehat{Z}^D_{t+1} + \beta\eta_{v^D G^D}\widehat{G}^D_{t+1} + \frac{1-\beta\phi_Z}{\phi_Z}\eta_{v^D Z^D}\widehat{Z}^D_{t+1} + \frac{1-\beta\phi_G}{\phi_G}\eta_{v^D G^D}\widehat{G}^D_{t+1}] + \eta_{v^D Z^D}\widehat{Z}^D_t + \eta_{v^D G^D}\widehat{G}^D_t =$

$= -[\frac{\beta\phi_Z + 1 - \beta\phi_Z}{\phi_Z}\eta_{v^D Z^D}\widehat{Z}^D_{t+1} + \frac{\beta\phi_G + 1 - \beta\phi_G}{\phi_G}\eta_{v^D G^D}\widehat{G}^D_{t+1}] + \eta_{v^D Z^D}\widehat{Z}^D_t + \eta_{v^D G^D}\widehat{G}^D_t$

$= -[\frac{1}{\phi_Z}\eta_{v^D Z^D}\widehat{Z}^D_{t+1} + \frac{1}{\phi_G}\eta_{v^D G^D}\widehat{G}^D_{t+1}] + \eta_{v^D Z^D}\widehat{Z}^D_t + \eta_{v^D G^D}\widehat{G}^D_t$

$= -\frac{\eta_{v^D Z^D}}{\phi_Z}(\widehat{Z}^D_{t+1} - \phi_Z\widehat{Z}^D_t) - \frac{\eta_{v^D G^D}}{\phi_G}(\widehat{G}^D_{t+1} - \phi_G\widehat{G}^D_t) =$

$= -\frac{\eta_{v^D Z^D}}{\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{\eta_{v^D G^D}}{\phi_G}\widehat{\xi}_{G^D t+1}$

$= -\frac{1}{\phi_Z}\frac{(1-\beta)\phi_Z(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{1}{\phi_G}\frac{(1-\beta)\phi_G(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

$= -\frac{(1-\beta)(\eta_{y^D Z^D} - \frac{1}{\sigma}\eta_{C^D Z^D})}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)(\eta_{y^D G^D} - \frac{1}{\sigma}\eta_{C^D G^D})}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

$= -\frac{(1-\beta)(\frac{(1-G)(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}} - \frac{1}{\sigma}\frac{(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}})}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)(\frac{G\frac{\varphi}{\sigma}}{1-G+\frac{\varphi}{\sigma}} - \frac{1}{\sigma}\frac{-G}{1-G+\frac{\varphi}{\sigma}})}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

$= -\frac{(1-\beta)\frac{\sigma(1-G)(1+\varphi)(1-\gamma) - (1+\varphi)(1-\gamma)}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)\frac{\sigma G\frac{\varphi}{\sigma} + G}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

$= -\frac{(1-\beta)\frac{(1+\varphi)(1-\gamma)[\sigma(1-G)-1]}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)\frac{G(\varphi+1)}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

$= \eta_{R^D\xi_{Z^D}}\widehat{\xi}_{Z^D t+1} - \eta_{R^D\xi_{G^D}}\widehat{\xi}_{G^D t+1}$

where we used $\widehat{Z}^D_t = \phi_Z\widehat{Z}^D_{t-1} + \widehat{\xi}_{Z^D t}$ and $\widehat{G}^D_t = \phi_G\widehat{G}^D_{t-1} + \widehat{\xi}_{G^D t}$.

Notice there is no α in these elasticities.

If $G = 0$: $\widehat{R}^D_{t+1} = -\frac{(1-\beta)(1+\varphi)(1-\gamma)(\sigma-1)}{(1-\beta\phi_Z)\sigma(1+\frac{\varphi}{\sigma})}\widehat{\xi}_{Z^D t+1}$

If $G = 0$ and $\gamma = 1$, $\widehat{R}^D_{t+1} = 0\ \forall\sigma,\varphi$

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If $G = 0$ and $\varphi = 0$, $\widehat{R}^D_{t+1} = -\frac{(1-\beta)(1-\gamma)(\sigma-1)}{(1-\beta\phi_Z)\sigma}\widehat{\xi}_{Z^D t+1}$

If $G \neq 0$ and $\gamma = 1$: $\widehat{R}^D_{t+1} = -\frac{(1-\beta)\frac{G(\varphi+1)}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$

If $G \neq 0$ and $\varphi = 0$: $\widehat{R}^D_{t+1} = -\frac{(1-\beta)(1-\gamma)[\sigma(1-G)-1]}{(1-\beta\phi_Z)\sigma(1-G)}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)G}{(1-\beta\phi_G)\sigma(1-G)}\widehat{\xi}_{G^D t+1}$

If $G \neq 0$, $\varphi = 0$ and $\sigma = 1$: $\widehat{R}^D_{t+1} = \frac{(1-\beta)(1-\gamma)G}{(1-\beta\phi_Z)(1-G)}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)G}{(1-\beta\phi_G)(1-G)}\widehat{\xi}_{G^D t+1}$

2.9 2nd-order approximation of the portfolio part of the model

From household FOCs for :
$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}R_{t+1}\}$, which can be written as: $\frac{C_t^{-\frac{1}{\sigma}}}{\beta} = E_t\{C_{t+1}^{-\frac{1}{\sigma}}R_{t+1}\}$
$C_t^{-\frac{1}{\sigma}} = \beta E_t\{C_{t+1}^{-\frac{1}{\sigma}}R^*_{t+1}\}$, which can be written as: $\frac{C_t^{-\frac{1}{\sigma}}}{\beta} = E_t\{C_{t+1}^{-\frac{1}{\sigma}}R^*_{t+1}\}$
Equating these two expressions gives us: $E_t(C_{t+1}^{-\frac{1}{\sigma}}R_{t+1}) = E_t(C_{t+1}^{-\frac{1}{\sigma}}R^*_{t+1})$, which can be written as $E_t(C_{t+1}^{-\frac{1}{\sigma}}R_{t+1}) - E_t(C_{t+1}^{-\frac{1}{\sigma}}R^*_{t+1}) = 0$

Take second-order approximation and evaluate it at steady state:
$E_t(-\frac{1}{\sigma}C_{t+1}^{-\frac{1}{\sigma}-1}dC_{t+1}R_{t+1}) + E_t(C_{t+1}^{-\frac{1}{\sigma}}dR_{t+1}) - E_t(-\frac{1}{\sigma}C_{t+1}^{-\frac{1}{\sigma}-1}dC_{t+1}R^*_{t+1}) - E_t(C_{t+1}^{-\frac{1}{\sigma}}dR^*_{t+1}) +$
$+ \frac{1}{2}[-\frac{1}{\sigma}(-\frac{1}{\sigma}-1)C_{t+1}^{-\frac{1}{\sigma}-1-1}d^2C_{t+1}R_{t+1} + C_{t+1}^{-\frac{1}{\sigma}}0 + 2(-\frac{1}{\sigma})C_{t+1}^{-\frac{1}{\sigma}-1}dC_{t+1}dR_{t+1}] -$
$- \frac{1}{2}[-\frac{1}{\sigma}(-\frac{1}{\sigma}-1)C_{t+1}^{-\frac{1}{\sigma}-1-1}d^2C_{t+1}R^*_{t+1} - C_{t+1}^{-\frac{1}{\sigma}}0 - 2(-\frac{1}{\sigma})C_{t+1}^{-\frac{1}{\sigma}-1}dC_{t+1}dR^*_{t+1}] =$
$= E_t(-\frac{1}{\sigma}C^{-\frac{1}{\sigma}-1}dC_{t+1}\frac{1}{\beta}) + E_t(C^{-\frac{1}{\sigma}}dR_{t+1}) - E_t(-\frac{1}{\sigma}C^{-\frac{1}{\sigma}-1}dC_{t+1}\frac{1}{\beta}) - E_t(C^{-\frac{1}{\sigma}}dR^*_{t+1}) +$
$+ \frac{1}{2}[-\frac{1}{\sigma}(-\frac{1}{\sigma}-1)C^{-\frac{1}{\sigma}-1-1}d^2C_{t+1}\frac{1}{\beta} + 2(-\frac{1}{\sigma})C^{-\frac{1}{\sigma}-1}dC_{t+1}dR_{t+1}] -$
$- \frac{1}{2}[-\frac{1}{\sigma}(-\frac{1}{\sigma}-1)C^{-\frac{1}{\sigma}-1-1}d^2C_{t+1}\frac{1}{\beta} + 2(-\frac{1}{\sigma})C^{-\frac{1}{\sigma}-1}dC_{t+1}dR^*_{t+1}] =$
$= E_t(C^{-\frac{1}{\sigma}}dR_{t+1}) - E_t(C^{-\frac{1}{\sigma}}dR^*_{t+1}) + \frac{1}{2}[2(-\frac{1}{\sigma})C^{-\frac{1}{\sigma}-1}dC_{t+1}dR_{t+1}] - \frac{1}{2}[2(-\frac{1}{\sigma})C^{-\frac{1}{\sigma}-1}dC_{t+1}dR^*_{t+1}]$
$= E_t(C^{-\frac{1}{\sigma}}dR_{t+1}) - E_t(C^{-\frac{1}{\sigma}}dR^*_{t+1}) + [-\frac{1}{\sigma}C^{-\frac{1}{\sigma}-1}dC_{t+1}dR_{t+1}] - [-\frac{1}{\sigma}C^{-\frac{1}{\sigma}-1}dC_{t+1}dR^*_{t+1}]$
Divide by $C^{-\frac{1}{\sigma}}$ and R.
$\widehat{R}_{t+1} - \widehat{R}^*_{t+1} + (-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}_{t+1}) - (-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}^*_{t+1}) = 0$

The same derivation for the foreign country gives:
$\widehat{R}^f_{t+1} - \widehat{R}^{f*}_{t+1} + (-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^f_{t+1}) - (-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^{f*}_{t+1}) = 0$

Subtract expressions for the home and foreign countries:
$\widehat{R}_{t+1} - \widehat{R}^*_{t+1} + (-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}_{t+1}) - (-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}^*_{t+1}) - [\widehat{R}^f_{t+1} - \widehat{R}^{f*}_{t+1} + (-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^f_{t+1}) - (-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^{f*}_{t+1})] = 0$
From Section 1.4: $R_{t+1} = \frac{Q_{t+1}}{Q_t}R^f_{t+1}$
Log-linearize: $\widehat{R}_{t+1} = \widehat{Q}_{t+1} - \widehat{Q}_t + \widehat{R}^f_{t+1}$
Same for the foreign:

Log-linearize: $\widehat{R}^*_{t+1} = \widehat{Q}_{t+1} - \widehat{Q}_t + \widehat{R}^{*f}_{t+1}$

Using this, simplify: $(-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}_{t+1}) - (-\frac{1}{\sigma}\widehat{C}_{t+1}\widehat{R}^*_{t+1}) - [-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^f_{t+1}) - (-\frac{1}{\sigma}\widehat{C}^{f*}_{t+1}\widehat{R}^{f*}_{t+1})] = 0$

$\widehat{C}_{t+1}\widehat{R}_{t+1} - \widehat{C}_{t+1}\widehat{R}^*_{t+1} - [\widehat{C}^{f*}_{t+1}\widehat{R}^f_{t+1} - \widehat{C}^{f*}_{t+1}\widehat{R}^{f*}_{t+1}] = 0$

$\widehat{C}_{t+1}\widehat{R}_{t+1} - \widehat{C}_{t+1}\widehat{R}^*_{t+1} - [\widehat{C}^{f*}_{t+1}(\widehat{R}_{t+1} - \widehat{Q}_{t+1} + \widehat{Q}_t) - \widehat{C}^{f*}_{t+1}(\widehat{R}^*_{t+1} - \widehat{Q}_{t+1} + \widehat{Q}_t)] = 0$

$\widehat{C}_{t+1}\widehat{R}_{t+1} - \widehat{C}_{t+1}\widehat{R}^*_{t+1} - [\widehat{C}^{f*}_{t+1}\widehat{R}_{t+1} - \widehat{C}^{f*}_{t+1}\widehat{R}^*_{t+1}] = 0$

$E_t(\widehat{C}^D_{t+1}\widehat{R}^D_{t+1}) = 0$

This results is the same as in GLR.

However, notice that there is no α in either expression:

Substitute expressions for $\widehat{C}^D_{t+1}$ from Section 2.2 (i.e., $\widehat{C}^D_{t+1} = \frac{(1+\varphi)(1-\gamma)}{1-G+\frac{\varphi}{\sigma}}\widehat{Z}^D_{t+1} - \frac{G}{1-G+\frac{\varphi}{\sigma}}\widehat{G}^D_{t+1}$) and $\widehat{R}^D_{t+1}$ from Section 2.8 (i.e., $\widehat{R}^D_{t+1} = -\frac{(1-\beta)\frac{(1+\varphi)(1-\gamma)[\sigma(1-G)-1]}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_Z}\widehat{\xi}_{Z^D t+1} - \frac{(1-\beta)\frac{G(\varphi+1)}{\sigma(1-G+\frac{\varphi}{\sigma})}}{1-\beta\phi_G}\widehat{\xi}_{G^D t+1}$).

Therefore, this will not pin down any α in contrast to GLR.

3 World Variables

GDPs of the two countries are given by:

$$y_t = RP_t Z_t L_t + RP_{*t} Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t}, \quad (1)$$

$$y_t^* = RP_t^* Z_t^{\gamma} Z_t^{*1-\gamma} L_t^* + RP_{*t}^* Z_t^* L_{*t}^*. \quad (2)$$

In log-linear form:

$$\hat{y}_t = L\left(\widehat{RP}_t + \hat{Z}_t + \hat{L}_t\right) + L_*\left[\widehat{RP}_{*t} + (1-\gamma)\hat{Z}_t + \gamma\hat{Z}_t^* + \hat{L}_{*t}\right],$$
$$\hat{y}_t^* = L^*\left[\widehat{RP}_t^* + \gamma\hat{Z}_t + (1-\gamma)\hat{Z}_t^* + \hat{L}_t^*\right] + L_*^*\left(\widehat{RP}_{*t}^* + \hat{Z}_t^* + \hat{L}_{*t}^*\right).$$

Equations (1)-(2) and symmetry of the steady state imply:

$$1 = L + L_* \quad \text{and} \quad 1 = L^* + L_*^*.$$

We will show below that the following results hold:

$$L = a, \quad L_* = 1-a, \quad L^* = a, \quad L_*^* = 1-a. \quad (3)$$

Hence,

$$\hat{y}_t = a\left(\widehat{RP}_t + \hat{Z}_t + \hat{L}_t\right) + (1-a)\left[\widehat{RP}_{*t} + (1-\gamma)\hat{Z}_t + \gamma\hat{Z}_t^* + \hat{L}_{*t}\right], \quad (4)$$

$$\hat{y}_t^* = a\left[\widehat{RP}_t^* + \gamma\hat{Z}_t + (1-\gamma)\hat{Z}_t^* + \hat{L}_t^*\right] + (1-a)\left(\widehat{RP}_{*t}^* + \hat{Z}_t^* + \hat{L}_{*t}^*\right). \quad (5)$$

Next, take a population-weighted average of equations (4) and (5), and define $\hat{y}_t^W$ as:

$$\begin{aligned}
\hat{y}_t^W &\equiv a\hat{y}_t + (1-a)\hat{y}_t^* = \\
&= a\left\{a\left(\widehat{RP}_t + \hat{Z}_t + \hat{L}_t\right) + (1-a)\left[\widehat{RP}_{*t} + (1-\gamma)\hat{Z}_t + \gamma\hat{Z}_t^* + \hat{L}_{*t}\right]\right\} \\
&\quad + (1-a)\left\{a\left[\widehat{RP}_t^* + \gamma\hat{Z}_t + (1-\gamma)\hat{Z}_t^* + \hat{L}_t^*\right] + (1-a)\left(\widehat{RP}_{*t}^* + \hat{Z}_t^* + \hat{L}_{*t}^*\right)\right\}. \quad (6)
\end{aligned}$$

Note that this can be rearranged as:

$$\begin{aligned}
\hat{y}_t^W &= a\left[a\widehat{RP}_t + (1-a)\widehat{RP}_{*t}\right] + (1-a)\left[a\widehat{RP}_t^* + (1-a)\widehat{RP}_{*t}^*\right] + \\
&\quad + aZ_t + (1-a)Z_t^* + a\left[a\widehat{L}_t + (1-a)\widehat{L}_{*t}\right] + (1-a)\left[a\widehat{L}_t^* + (1-a)\widehat{L}_{*t}^*\right].
\end{aligned}$$

Note also that labor market clearing in the two countries requires: $a\widehat{L}_t + (1-a)\widehat{L}_{*t} = \hat{L}_t^S$ and $a\widehat{L}_t^* + (1-a)\widehat{L}_{*t}^* = \widehat{L}_t^{*S}$, where S superscripts denote the total amounts of labor supplied in the two countries. Defining $\hat{Z}_t^W \equiv a\hat{Z}_t + (1-a)\hat{Z}_t^*$ and $\hat{L}_t^W \equiv a\hat{L}_t^S + (1-a)\hat{L}_t^{*S}$ makes it possible to write:

$$\hat{y}_t^W = a\left[a\widehat{RP}_t + (1-a)\widehat{RP}_{*t}\right] + (1-a)\left[a\widehat{RP}_t^* + (1-a)\widehat{RP}_{*t}^*\right] + \hat{Z}_t^W + \hat{L}_t^W. \quad (7)$$

Labor demand equations by firms in the two countries and production functions for domestic and offshored production imply:

$$\begin{aligned}
RP_t Z_t L_t &= aRP_t^{1-\omega}(C_t + G_t), \\
RP_{*t} Z_t^{1-\gamma} Z_t^{*\gamma} L_{*t} &= (1-a) RP_{*t}^{1-\omega}(C_t + G_t), \\
RP_t^* Z_t^{\gamma} Z_t^{*1-\gamma} L_t^* &= aRP_t^{*1-\omega}(C_t^* + G_t^*), \\
RP_{*t}^* Z_t^* L_{*t}^* &= (1-a) RP_{*t}^{*1-\omega}(C_t^* + G_t^*).
\end{aligned}$$

In the symmetric steady state, these equations imply the results (3). Moreover, from these equations, GDP expressions, and the definition of $\hat{y}_t^W$, it follows that:

$$\hat{y}_t^W = (1-G)\hat{C}_t^W + G\hat{G}_t^W + a\left[a\widehat{RP}_t + (1-a)\widehat{RP}_{*t}\right] + (1-a)\left[a\widehat{RP}_t^* + (1-a)\widehat{RP}_{*t}^*\right], \quad (8)$$

where we defined $\hat{C}_t^W \equiv a\hat{C}_t + (1-a)\hat{C}_t^*$ and $\hat{G}_t^W \equiv a\hat{G}_t + (1-a)\hat{G}_t^*$.

Next, observe that aggregate per capita demand of consumption output in the two countries is equal to $y_t^D \equiv C_t + G_t$ and $y_t^D \equiv C_t^* + G_t^*$, or, in log-linear terms:

$$\hat{y}_t^D = (1-G)\hat{C}_t + G\hat{G}_t \quad \text{and} \quad \hat{y}_t^{*D} = (1-G)\hat{C}_t^* + G\hat{G}_t^*.$$

Taking a population-weighted average of these equations defines $\hat{y}_t^{WD} \equiv (1-G)\hat{C}_t^W + G\hat{G}_t^W$. This and equation (8) together imply:

$$a\left[a\widehat{RP}_t + (1-a)\widehat{RP}_{*t}\right] + (1-a)\left[a\widehat{RP}_t^* + (1-a)\widehat{RP}_{*t}^*\right] = 0, \tag{9}$$

and:

$$\hat{y}_t^W = \hat{Z}_t^W + \hat{L}_t^W. \tag{10}$$

Note that equilibrium in the world market for consumption requires: $\hat{y}_t^W = \hat{y}_t^{WD}$, or:

$$\hat{y}_t^W = (1-G)\hat{C}_t^W + G\hat{G}_t^W. \tag{11}$$

Log-linear versions of optimal price setting equations imply:

$$\widehat{RP}_t = \hat{w}_t - \hat{Z}_t, \tag{12}$$

$$\widehat{RP}_t^* = \hat{w}_t^* - \left[\gamma\hat{Z}_t + (1-\gamma)\hat{Z}_t^*\right], \tag{13}$$

$$\widehat{RP}_{*t} = \hat{w}_t - \left[(1-\gamma)\hat{Z}_t + \gamma\hat{Z}_t^*\right], \tag{14}$$

$$\widehat{RP}_{*t}^* = \hat{w}_t^* - \hat{Z}_t^*, \tag{15}$$

and substituting these into (9) yields:

$$\hat{w}_t^W = \hat{Z}_t^W, \tag{16}$$

where we defined $\hat{w}_t^W \equiv a\hat{w}_t + (1-a)\hat{w}_t^*$.

Finally, taking a population-weighted average of the log-linear versions of home and foreign labor supply equations yields:

$$\hat{L}_t^W = -\frac{\varphi}{\sigma}\hat{C}_t^W + \varphi\hat{w}_t^W. \tag{17}$$

The system of equations (10), (11), (16), and (17) determines the endogenous variables $\hat{y}_t^W$, $\hat{L}_t^W$, $\hat{C}_t^W$, and $\hat{w}_t^W$ as functions of the exogenous shocks $\hat{G}_t^W$ and $\hat{Z}_t^W$. It is easy to verify that this is the same system of equations as in GLR. It follows that the change in production structure and demand-fulfillment from the one in GLR to the one we are studying in this paper matters for how world production is allocated between the two countries but not for the overall amount of world production.

References

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