Scattered spaces in Galois geometry

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Scattered spaces

- In topology: A space W is scattered if every non-empty subset T contains a point isolated in T. [Cantor 1872]

- In projective geometry: W is scattered w.r.t. a set of subspaces D, if W intersects each element of D in at most a point. (Motivated by the theory of blocking sets in projective spaces over finite fields [Ball-Blokhuis-ML 2000])

NOTATION AND TERMINOLOGY

Vector space <-> Projective space

$V=V(n,q)$ <-> $PG(V)=PG(n-1,q)$

subspace U <-> subspace $PG(U)$

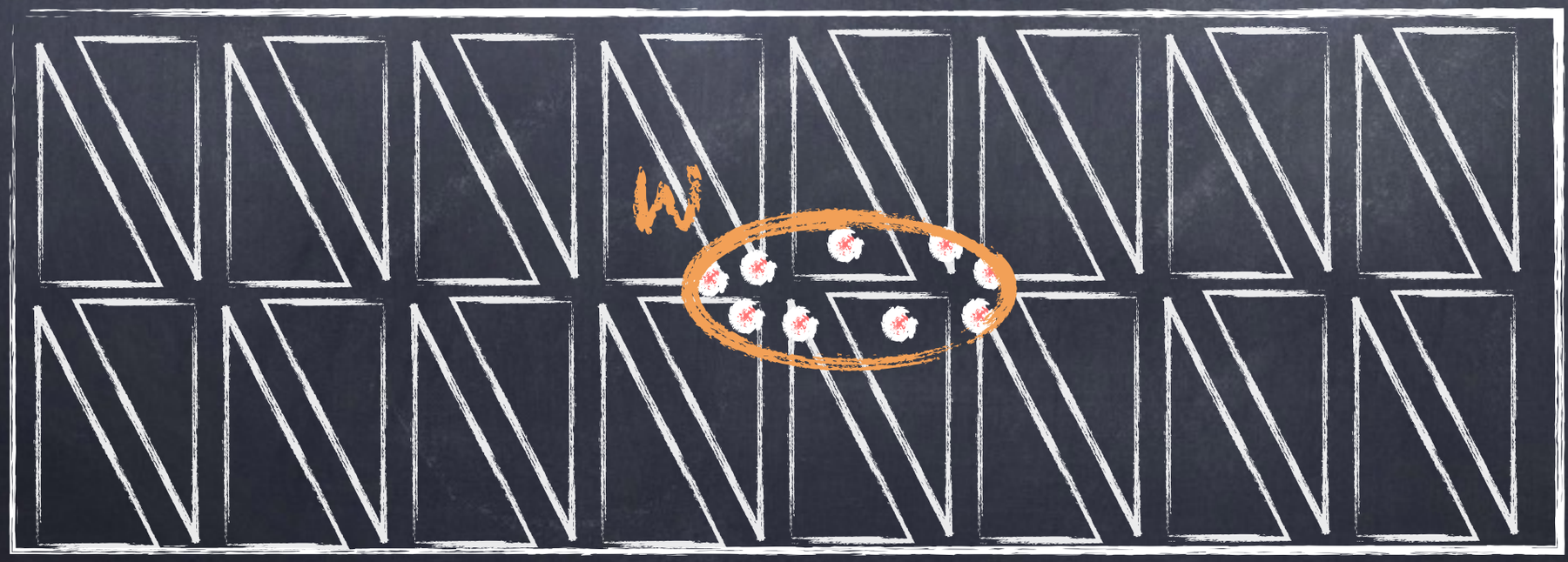
$\dim(U) = \dim(PG(U)) + 1$

Scattered spaces in Galois geometry

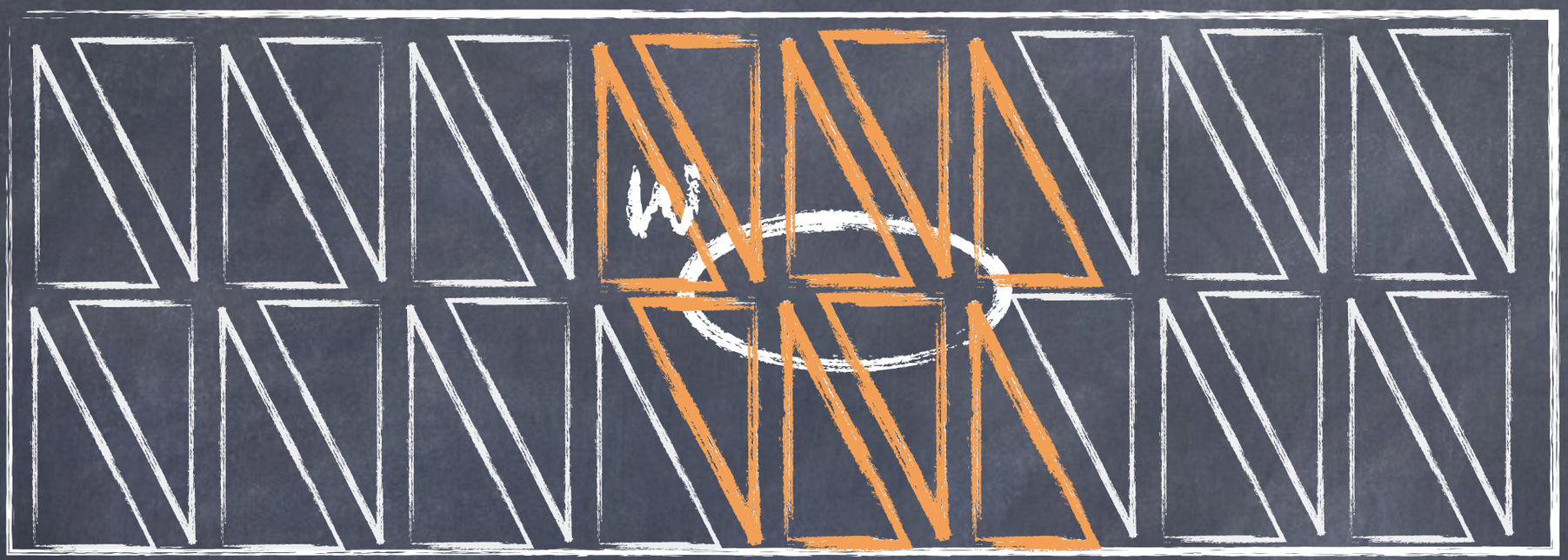
- D= any set of subspaces in V(n,q).
- A subspace W is scattered w.r.t. D <=> $\forall R \in D$: $\dim(W \cap R) \leq 1$
- In this talk D will usually be a spread

Partition of V\{0} by subspaces of constant dimension

Projective space picture: PG(n-1,q)



Let $\mathcal{D}$ be a spread of $PG(n-1,q)$



$PG(n-1,q)$

$B(W):=\{R \text{ in } \mathcal{D} : \dim(W \cap R) \geq 1\}$

If W is scattered and $\dim W = d$ then $|B(W)|=(q^d-1)/(q-1)$

Recent work related to scattered spaces

(in alphabetical order)

S. Ball, S. Barwick, A. Blokhuis, B. Csajbók,
G. Donati, N. Durante, W. Jackson, G. Lunardon,
G. Marino, O. Polverino, R. Trombetti,
J. Sheekey, G. Van de Voorde, C. Zanella

Outline

PART I

Basic theory and constructions

PART II

Applications

PART II Applications (a teaser)

1. Translation hyperovals
2. Field reduction and linear set
3. Two-intersection sets
4. Two-weight codes
5. Blocking sets
6. Embeddings of Segre varieties
7. Pseudo-reguli
8. Splashes of subgeometries
9. MRD codes
10. Semifield theory

PART I Basic theory and constructions

References

[1] S. Ball; A. Blokhuis; M. Lavrauw: Linear (q+1)-fold blocking sets in PG(2,q^4). Finite Field Appl. 2000

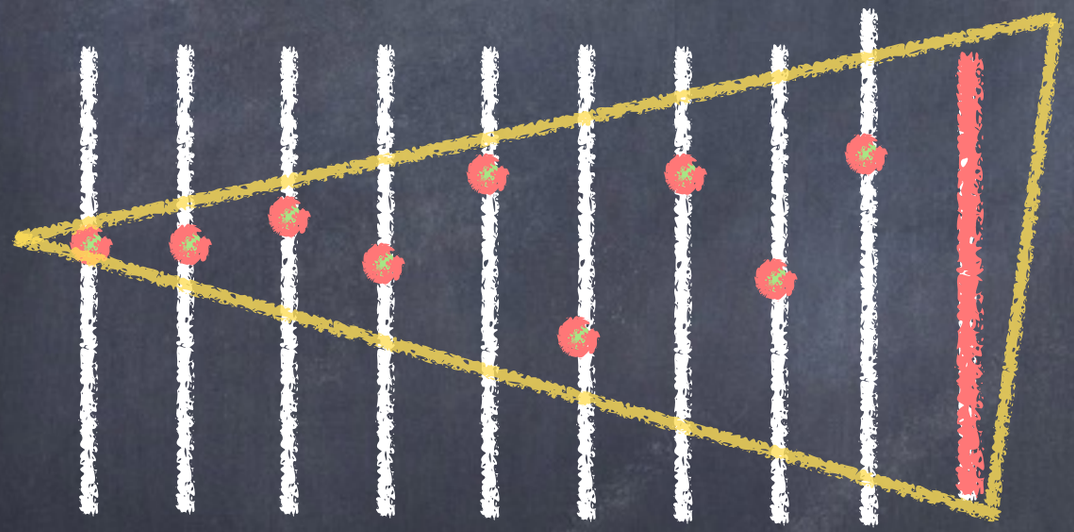
[2] A. Blokhuis; M. Lavrauw: Scattered spaces with respect to a spread in PG(n,q). Geom. Dedicata, 2000

[3] M. Lavrauw: Scattered spaces with respect to spreads and eggs in finite projective spaces. Dissertation, 2001.

1.1 First example: Line spread D in PG(3,q)

(a) every line not contained in D is scattered;

(b) no plane is scattered

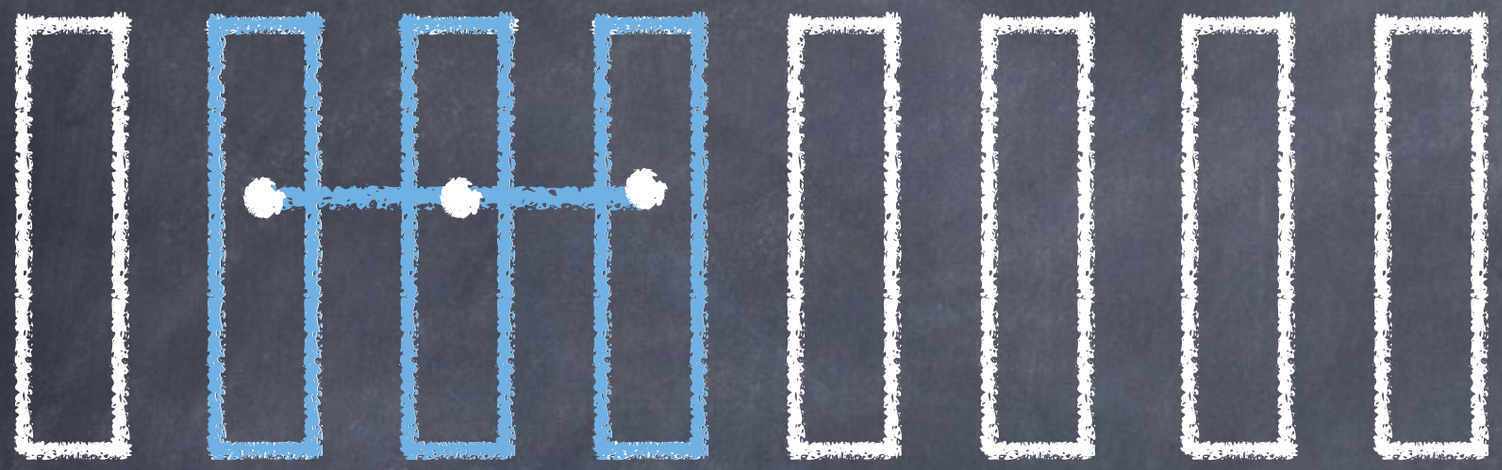


q^2+1 lines

q^2+q+1 points

(a),(b) => scattered space has proj. dim. ≤ 1

1.2 Second example: plane spread in PG(5,q)



q^3+1

planes

(a) every line not contained in an element of S is scattered;

(b) no solid is scattered, since $|\mathcal{D}|=q^3+1$;

(c) does there exist a scattered plane?

Maximal and maximum scattered

Let U be scattered space w.r.t. D

- U is called **maximal** scattered if U is not contained in a larger scattered space
- U is called **maximum** scattered if any scattered space T w.r.t. D has $\dim T \leq \dim U$

1.3. Lower bound on maximal scattered spaces

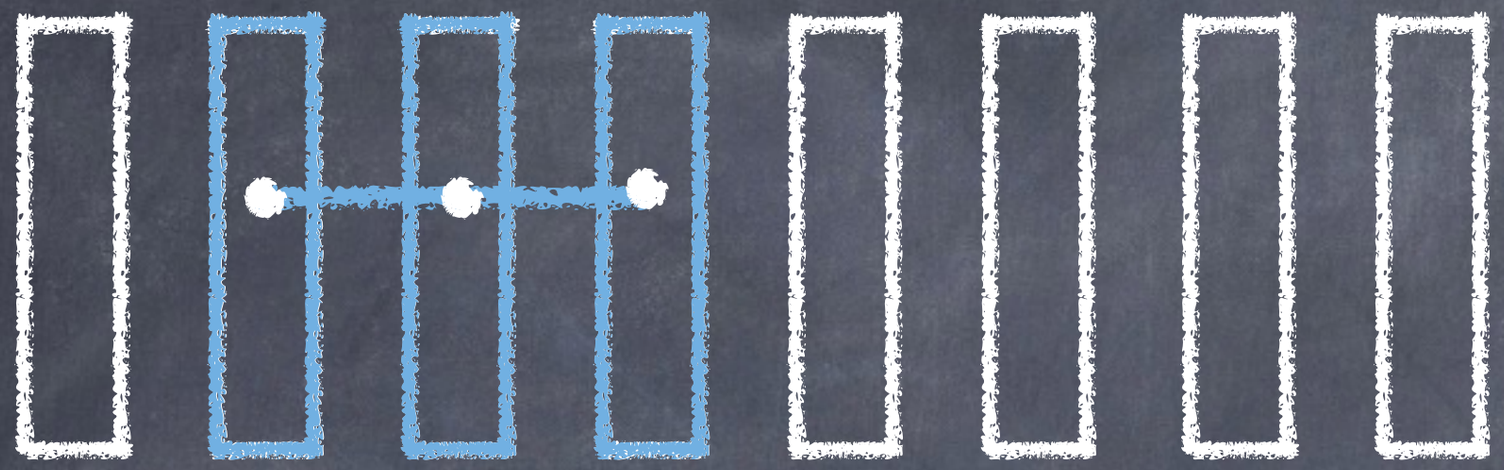
Theorem 1 If U is maximal scattered w.r.t. a t-spread in V(rt,q), then dim U ≥ (rt-t)/2 + 1.

Proof If U is scattered, then <U,x> is NOT scattered iff x is contained in some <R,U> with R∈B(U). Counting all such x gives (m=dimU)

$(q^{m+t-1}-q^m)(q^{m-1}+q^{m-2}+q^{m-3}+\ldots+q+1)+q^m$

which should then be compared to the $|V(rt,q)|=q^{rt}$. □

Back to a plane spread in PG(5,q)



q^3+1

planes

Suppose U in V(6,q) is maximal scattered. Then

(a) dim U ≥ 2

(b) dim U < 4

(c) Theorem 1 (r=2,t=3) => dim U ≥ 3/2+1 => dim U ≥ 3

(a),(b),(c) => PG(U) is plane in PG(5,q)

NOTE: maximum ≈ maximal in this case

1.4 Upper bound on scattered spaces

Theorem 2 If U is scattered w.r.t. a t-spread in V(rt,q), then dim U ≤ rt-t.

REMARK: The bound from Theorem 2 is sharp -> "scattering spreads"

1.5 Changing the problem: "scattering spreads"

Given a subspace U, can we find a spread S, such that U is scattered w.r.t. S?

Theorem 3 If $\dim U \leq rt-t$, then there exists a spread S such that U is scattered w.r.t. S.

Theorem 3 => bound from Theorem 2 is sharp.

1.6 Desarguesian spreads

- Spread S -> $\Omega(S)$ design with parallelism

- S is a Desarguesian spread if $\Omega(S)$ is a Desarguesian affine space

- $D_{r,t,q}$ = Desarguesian t-spread in $V(rt,q)$

1.6 Desarguesian spreads

Theorem 4 If U is scattered w.r.t. $D_{r,t,q}$ then dim U ≤ rt/2.

Corollary 5 A maximal scattered space U w.r.t $D_{r,t,q}$ satisfies (rt-t)/2 + 1 ≤ dim U ≤ rt/2.

Corollary 6 If t=2, then a maximal scattered space w.r.t. $D_{r,t,q}$ is maximum scattered and has dim = r.

1.6 Desarguesian spreads

Theorem 7 If r is even, then the upper bound rt/2 is sharp.

Lower bound for a maximum scattered space:

Theorem 8 The dimension of a maximum scattered space w.r.t. $D_{r,t,q}$ is $\geq r'k$ where $r'|r$, $(r',t)=1$, and $r'k$ is maximal such that

$$k < (rt-t+3)/2 \text{ for } q=2 \text{ and } r'=1$$

and

$$r'k < (rt-t+r'+3)/2 \text{ otherwise.}$$

1.7. Many open problems!

1. Determine the exact dimension for maximum scattered spaces w.r.t. $D_{r,t,q}$ (for r odd).

2. What is the minimum dimension of maximal scattered spaces w.r.t. $D_{r,t,q}$?

3. What is the dimension of the second largest maximal scattered spaces w.r.t. $D_{r,t,q}$?

4. Can we determine the spectrum of maximal scattered spaces w.r.t. $D_{r,t,q}$?

5. Constructions! (geometric, polynomial method, tensor products, ...)

PART II Applications

1. Translation hyperovals
2. Field reduction and linear sets
3. Two-intersection sets
4. Two-weight codes
5. Blocking sets
6. Embeddings of Segre varieties
7. Pseudo-reguli
8. Splashes of subgeometries
9. MRD codes
10. Semifield theory

1. Translation hyperovals

Scattered t-space in V(2t,2) w.r.t t-spread S <-->
translation hyperoval in the projective plane ABB(S)
(André-Bruck-Bose)

[Denniston 1979], [O'Keefe-Pascasio-Penttila 1992], [Jha-Johnson 1992], [Cherowitzo 2010]

2. Field reduction and linear sets

Field reduction map $F_{r,t,q}$: $V(r,q^t)\to V(rt,q)$

- $D_{r,t,q}=\{F_{r,t,q}(\langle v\rangle): v\in V\setminus\{0\}\}$ is a Desarg. t-spread
- If $U\subseteq V(rt,q)$ => $B(U)\subseteq PG(r-1,q^t)$
- $B(U)$ is called an F_q-linear set
- dim U = rank of the linear set $B(U)$

Proposition 9 An F_q-linear set $B(U)$ in $PG(r-1,q^t)$ has maximal size (w.r.t. to its rank) iff U is scattered w.r.t. $D_{r,t,q}$

(more on this by B. Csajbók this afternoon)

3. Two-intersection sets

W.r.t. hyperplanes in $PG(r-1,q^t)$:

- Type 1: if $t=2u$, union of $PG(r-1,q^u)$'s
- Type 2: if $r=2s$, then union of $(s-1)$-dim subspaces

Theorem 10 [Blokhuis-ML 2000] If U is scattered w.r.t. $D_{r,t,q}$ of dim $rt/2$ then $B(U)$ is a two-intersection set w.r.t. hyperplanes in $PG(r-1,q^t)$, projectively inequivalent to the sets of type 1, 2.

two-intersection sets "of scattered type"

4. Two-weight codes

Each two-intersection set gives a two-weight code.

Theorem 11 [Blokhuis-ML 2000] A scattered space of dimension $m=rt/2$ w.r.t. $D_{r,t,q}$ gives rise to a linear $[(q^m-1)/(q-1),r]$-code with weights

$$q^{m-t}(q^t-1)/(q-1) \text{ and } q^{m-t+1}(q^{t-1}-1)/(q-1).$$

It follows from Theorem 10 that scattered two-weight codes are inequivalent to the ones arising from sets of type 1 and 2.

5. Blocking sets

[Blokhuis, Storme and Szönyi 1999] showed that an s-fold blocking set in $PG(2,q^4)$ of size $s(q^4+1)+c$ with s and c small enough contains the union of s disjoint Baer subplanes.

In [Ball-Blokhuis-ML 2000] a scattered linear set of rank 6 is constructed, which gives a $(q+1)$-fold blocking set of size $(q+1)(q^4+q^2+1)$ in $PG(2,q^4)$ $(r=3,t=4)$ which is not the union of Baer subplanes.

5. Blocking sets

Theorem 12 [Blokhuis-ML 2000] A scattered subspace W of dimension m w.r.t. $D_{r,t,q}$ induces a $(\mu_{k-1}(q))$-fold blocking set B(W) w.r.t. $((rt-m+k)/t-1)$-dimensional subspaces in $PG(r-1,q^t)$, of size $\mu_{m-1}(q)$ where $1\le k\le m$ and $t \mid m-k$.

Previous result [Ball-Blokhuis-ML 2000] for m=6, r=3, t=4, k=2:

$\mu_{k-1}(q)= \mu_1(q)= q+1$

$(rt-m+k)/t-1= (12-6+2)/4-1= 1$

$PG(r-1,q^t)= PG(2,q^4)$

6. Embeddings of Segre varieties

Theorem 13 [ML-Sheekey-Zanella 2014]

If U is a maximum scattered subspace w.r.t. $D_{2,t,q}$ then $B(U)$ (in $PG(2t-1,q)$) is a minimum embedding of the Segre variety $S_{t,t}(q)$.

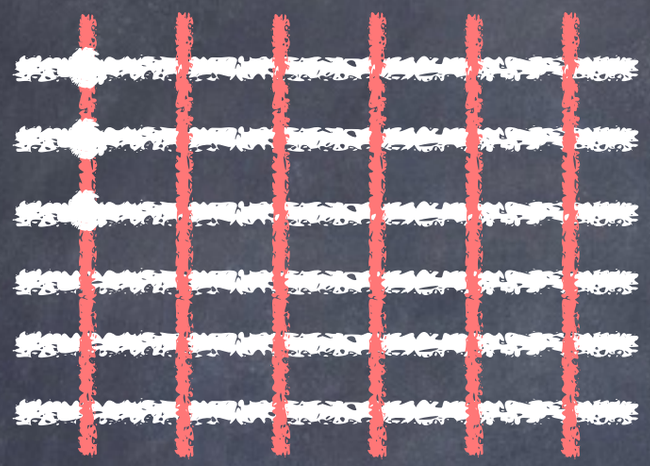
(natural embedding is in $PG(t^2-1,q)$)

This minimum embedding turns out to be quite an interesting hypersurface

7. Pseudo-regulus

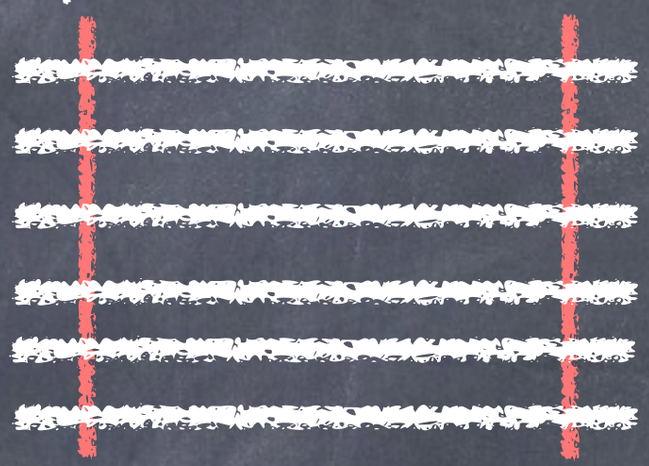
Generalizes the concept of a regulus in PG(3,q)

regulus



q+1 lines

pseudoregulus



q+1 transversals

2 transversals

- [Freeman 1980] PG(3,q^2): rt/2=4
- [Marino et al. 2007] PG(3,q^3): rt/2=6
- [ML-Van de Voorde 2013] PG(2n-1,q^3): rt/2=3n
- [Lunardon et al. 2014] PG(2n-1,q^t): rt/2=nt

8. Splashes of subgeometries

PG(2,q^t)

[Barwick-Jackson 2014, 201*] PG(2,q^3)

[ML-Zanella 2015] PG(r-1,q^t) + characterisation of scattered splashes

9. MRD codes

(see talk by J. Sheekey on Friday)

Scattered space of dimension n in V(2n,q) -> MRD code of dimension 2n over GF(q), and minimum distance n-1.

-> U_f : (x,f(x)) and D_a : (x,ax) (x in $GF(q^n)$)

-> $U_f \cap D_a$: $f(x)=ax \iff (f-a)(x)=0$

-> U_f scattered => $\ker(f-a)\leq 1$ for all $0\neq a$ in $GF(q^n)$

-> $\text{rank}(a+bf) \geq n-1$, for all a,b in $GF(q^n)$, $(a,b)\neq(0,0)$

-> $C_f = \langle 1,f \rangle$ is an MRD code

10. Semifield theory

Finite non-associative division algebras: same axioms as for a finite field, but we do NOT assume commutativity and associativity for multiplication.

First studied by L. E. Dickson (1906)

Example: Generalized Twisted Fields (GTF) (Albert 1960)

$$(GF(q^n),+,*): \quad x*y = xy - cx^a y^b$$

where $Fix(a)=Fix(b)=F_q$, and $N(c)\neq 1$

A recent construction

[Dempwolff 2013]: translation planes and semifields from Dembowski-Ostrom polynomials

$\mathbb{S}(F_1, F_2) = (\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}, +, *)$, whose multiplication is given by

$$(u, v) * (x, y) = (u, v)\begin{pmatrix} x & y \\ F_1(y) & \xi F_2(x) \end{pmatrix}, \qquad (*)$$

A) $F_1(x) = F_2(x) = A_{a,r}(x) = x^{q^r} - ax^{q^{-r}}$ such that $gcd(n, r) = 1$ and a is an element of $\mathbb{F}_{q^n}^*$ with $N_q(a) \neq 1$, where $N_q : \mathbb{F}_{q^n} \to \mathbb{F}_q$ denotes the norm function from $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$;

B) $F_1(x) = F_2(x) = B_{b,r}(x) = 2H_{b,r}^{-1}(x) - x$ such that $H_{b,r}(x) = x - bx^{q^r}$, $gcd(n, r) = 1$ and $b \in \mathbb{F}_{q^n}^*$ with $N_q(b) \neq \pm 1$;

AB) $F_i(x) = A_{b^2,r}(x)$, $F_j(x) = B_{b,-r}(x)$, $\{i, j\} = \{1, 2\}$ such that $gcd(n, r) = 1$ and $N_q(b) \neq \pm 1$.

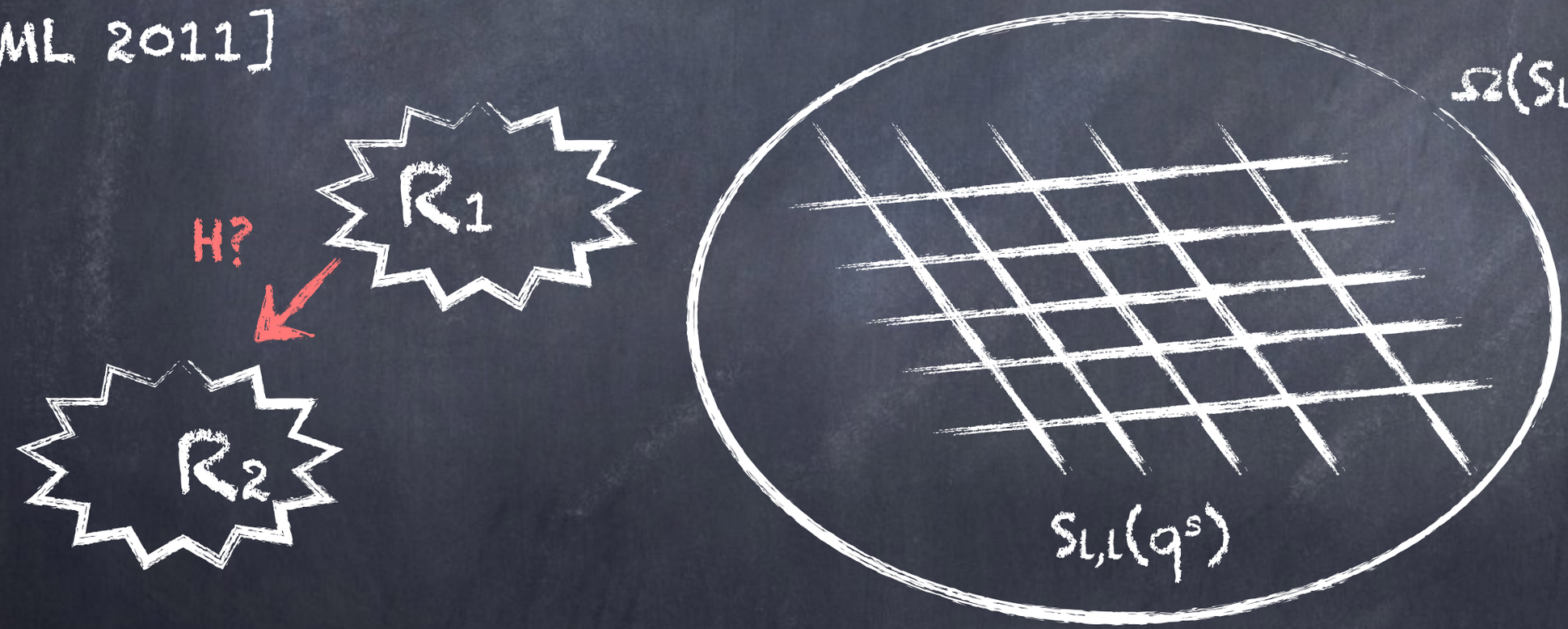
Scattered linear sets and semifields

- Semifield $, l-dim over $N_l(\$)$, ls-dimensional over its center $Z(\$)=GF(q)$

-> linear set $R(\$)$ of rank ls, disjoint from $\Omega(S_{l,l}(q^s))$

-> isotopism class $[\$]$ <-> orbit of $H=H(S_{l,l}(q^s))$

[ML 2011]



-> $ is a scattered semifield <=> $R(\$)$ is scattered

Scattered linear sets and semifields

$(u,v)*(x,y)=(ux+vF_1(y),uy+v\mu F_2(x))$ (μ a nonsquare in GF(q))

A) $F_1(x) = F_2(x) = A_{a,r}(x) = x^{q^r} - ax^{q^{-r}}$ such that $gcd(n, r) = 1$ and a is an element of $\mathbb{F}_{q^n}^*$ with $N_q(a) \neq 1$, where $N_q : \mathbb{F}_{q^n} \to \mathbb{F}_q$ denotes the norm function from $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$;

B) $F_1(x) = F_2(x) = B_{b,r}(x) = 2H_{b,r}^{-1}(x) - x$ such that $H_{b,r}(x) = x - bx^{q^r}$, $gcd(n, r) = 1$ and $b \in \mathbb{F}_{q^n}^*$ with $N_q(b) \neq \pm 1$;

AB) $F_i(x) = A_{b^2,r}(x)$, $F_j(x) = B_{b,-r}(x)$, $\{i, j\} = \{1, 2\}$ such that $gcd(n, r) = 1$ and $N_q(b) \neq \pm 1$.

Theorem 14 [ML-Marino-Polverino-Trombetti 2014, 2015]

A) contains new* semifields

B) always isotopic to GTF : $x*y = xy - cx^ay^b$

AB) contains new* semifields

*including Knuth operations and extension

Thank you for your attention!