A Survey of Graphs
Hamiltonian–Connected from a Vertex

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ABSTRACT

A graph $G$ is called hamiltonian–connected from a vertex $v$ if a hamiltonian path exists from $v$ to every other vertex $w \neq v$. We present a survey of the main results known about such graphs, including a section on graphs uniquely hamiltonian–connected from a vertex and a section on the computational complexity of determining whether a given graph is hamiltonian–connected from a vertex or uniquely hamiltonian–connected from a vertex.

1. Introduction

In 1963, Ore [19] defined a graph to be hamiltonian–connected if there is a hamiltonian path between every pair of distinct vertices. Since that time, a great deal of work has been done on this subject (see, for example, [2], [23], [25], [4]). At the Fourth International Conference on the Theory and Applications of Graphs in 1980, Chartrand and Nordhaus [3] extended this concept by defining a graph to be hamiltonian–connected from a vertex $v$ if there is a hamiltonian path from a distinguished vertex $v$ to every other vertex. This generalization arises quite naturally in many situations. For example, one might consider the problem of a communications network in which all communications arise at a central office, are routed through all offices and end at any one of the other offices, or vice versa. (In addition, one of the authors is particularly fond of the "Halloween party" example, in which the participants start at their respective houses, trick–or–treat at all other houses, and end at the
Examples of graphs hamiltonian-connected from a vertex $v$ are given in Figure 1.

We use $HC_v$ to denote hamiltonian-connected from a vertex $v$ (and, similarly, $HC$ to denote hamiltonian-connected). We use $p$ to denote the order of a graph, and we denote the set of all neighbors of a vertex $u$ by $N(u)$.

One of the most interesting problems in the study of graphs hamiltonian-connected from a vertex has been to determine the minimum number of edges for such a graph. G.R.T. Hendry originally raised the possibility that this minimum number of edges would be attained by graphs with the minimum number of hamiltonian paths originating at $v$. As we will see, Hendry went on to produce the surprising result that this is false. However, it was this idea that led him to make the following definition in 1984 ([8]): A
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A graph is called *uniquely hamiltonian-connected from a vertex* \( v \) (UHC\( v \)) if there is a unique hamiltonian path from \( v \) to every other vertex. In Figure 1, graphs \( G_2 \) and \( G_3 \) are uniquely hamiltonian-connected from \( v \), while \( G_1 \) and \( G_4 \) each have multiple paths from \( v \) to at least one other vertex.

The diagram in Figure 2 is taken from [8]. It is clear that the set of graphs UHC\( v \) and the set of graphs HC are both properly contained in the set of graphs HC\( v \). The intersection of UHC\( v \) and HC was shown by Hendry in [8] to consist solely of the set \( \{K_1, K_2, K_3\} \). (Similarly, the obvious definition of graphs "uniquely hamiltonian-connected" yields only these three graphs.) Finally, it is shown in [3] by Chartrand and Nordhaus that the set of graphs HC\( v \) is properly contained in the set of all hamiltonian graphs.

In the next section, we present some general results about graphs hamiltonian-connected from a vertex. In Section 3, a survey of graphs uniquely hamiltonian-connected from a vertex is given. We then return to the motivating problem of minimizing the number of edges in graphs HC\( v \) in Section 4. In Section 5, we discuss the computational complexity of HC\( v \) and of UHC\( v \), and we conclude with a section discussing possible avenues for further exploration.
2. General Results on Graphs HCv

In the first paper on the subject of graphs hamiltonian–connected from a vertex, Chartrand and Nordhaus [3] presented the following results:

**Theorem 1** (Chartrand and Nordhaus, [3]) Let $G$ be a graph of order $p$ with $p \geq 4$.
1. If $G$ is HCv, then both $G$ and $G - v$ are 2-connected.
2. If $G$ is HCv with $u \in N(v)$, then $\deg(u) \geq 3$.
3. If $G$ contains a vertex adjacent to two vertices $u$ and $v$ each of degree 2, then $G$ is hamiltonian-connected from at most two vertices, namely $u$ and $v$.

The graph $G_2$ in Figure 1 is hamiltonian-connected from exactly one vertex, namely $v$. The complete graph $K_p$ is clearly hamiltonian-connected from every vertex. It is easily seen that a graph cannot be hamiltonian-connected from exactly $p - 1$ vertices. Chartrand and Nordhaus gave a class of examples in [3] illustrating that all other possibilities are realized: a graph $G$ of order $p$ may be hamiltonian-connected from any of 0, 1, 2, ..., $p - 2$, $p$ vertices.

Ore proved in [18] that if a graph $G$ with order $p \geq 2$ satisfies $\deg x + \deg y \geq p + 1$ for all pairs $x, y$ of nonadjacent vertices, then $G$ is hamiltonian-connected. Chartrand and Nordhaus proved a similar result in [3] to obtain the following sufficient condition for a graph to be hamiltonian-connected from a vertex. Examples were given in [3] to show that this result is best possible.

**Theorem 2** (Chartrand & Nordhaus, [3]) Let $G$ be a hamiltonian graph of order $p$. If there exists a vertex $u$ such that $\deg u + \deg w \geq p + 1$ for each vertex $w$ not adjacent to $u$, then $G$ is hamiltonian-connected from at least two vertices. (In particular, $G$ is hamiltonian-connected from the two vertices adjacent to $u$ on a hamiltonian cycle.)

**Theorem 3** (Chartrand & Nordhaus, [3]) Let $G$ be a hamiltonian graph of order $p$ and let $C: v_1, v_2, ..., v_p, v_1$ be a hamiltonian cycle of $G$. If $v_1$ and $v_j$ ($j \neq 1$) are nonadjacent and $\deg v_1 + \deg v_j \geq p + 1$, then $G$ is hamiltonian-connected from $v_p$ and $v_2$ if and only if $G + v_1v_j$ is hamiltonian-connected from $v_p$ and $v_2$.

**Theorem 4** (Chartrand & Nordhaus, [3]) Let $G$ be a hamiltonian graph of order $p$. Suppose there exists a vertex $u$ such that $\deg u + \deg v \geq p$ for each vertex $v$ not adjacent with $u$. Define

$$S = \{v : v \neq u, uv \notin E(G), \deg u + \deg v = p\}.$$
Then \( G \) contains at least two vertices, namely those consecutive with \( u \) on a hamiltonian cycle, from which hamiltonian paths exist to all other vertices with at most \( |S| \) exceptions.

3. Graphs Uniquely Hamiltonian-Connected from a Vertex

The study of graphs uniquely hamiltonian-connected from a vertex has yielded some of the richest results in the field. It is hoped that the methods and ideas used in the study of these graphs will enable us to better understand the larger class of all graphs hamiltonian-connected from a vertex. As mentioned above, the complete graphs \( K_1, K_2, \) and \( K_3 \) are uniquely hamiltonian-connected from a vertex. These are trivial cases, however, and in all that follows we assume that the order of the graph is greater than 3.

The following theorem gives some of the major results from [8] and [10].

**Theorem 5** (Hendry, [8, 10]) Let \( G \) be a graph UHC\( v \). Then:
1. \( \deg v \) is even.
2. \( G \) has \( \frac{\deg v}{2} \) hamiltonian cycles.
3. \( p \) is odd.
4. \( |E| = \frac{3(p - 1)}{2} \)
5. \( G - v \) has a unique hamiltonian cycle.
6. Every vertex other than \( v \) has degree 2, 3, or 4.
   If \( u \in N(v) \), then \( \deg u = 3 \).
7. Every edge of \( G \) lies on at least 2 hamiltonian paths from \( v \).
   Furthermore, every edge is traversed at least once in each direction by some hamiltonian path from \( v \).

A forced edge is defined to be an edge which lies on every hamiltonian path from \( v \). Hendry made many observations about forced edges in [9]; for example, he notes that no forced edge can lie on a triangle.

The definition of forced edge, given for graphs UHC\( v \), applies equally well to graphs HC\( v \). The complete graphs \( K_p \) give examples of graphs HC\( v \) which contain no forced edges. However, every graph UHC\( v \) must contain at least one forced edge. Indeed, the following theorem gives sharp upper and lower bounds for the number of forced edges in a graph UHC\( v \).

**Theorem 6** (Hendry, [8]) Let \( G \) be UHC\( v \). Let \( n_2 \) denote the number of vertices of degree 2 in \( G \). Then the number, \( f_G \), of forced edges in \( G \) satisfies:
\[
n_2/2 \leq f_G \leq (p - 1 - \deg v)/2.
\]
If $G$ is a graph $UHCv$, and $u$ is a vertex in $G - v$, we use the notation $H_u$ to denote the hamiltonian path from $v$ to $u$. We say vertex $x$ is penultimate to vertex $y$ if $x$ is adjacent to $y$ in $H_y$.

**Theorem 7** (Hendry, [10]) Let $G$ be $UHCv$. Let $x$ be a vertex with $x \neq v$. Then:
1. if $\deg x = 2$, then $x$ is penultimate to both its neighbors,
2. if $\deg x = 4$ or if $x \in N(v)$, then $x$ is not penultimate to any neighbor,
3. if $\deg x = 3$ with $x \notin N(v)$, then $x$ is penultimate to exactly one of its neighbors.

The next result relates the number of vertices of different degrees in a graph $UHCv$. Part (1) follows from counting vertices, part (2) follows from the previous theorem by counting penultimate vertices in hamiltonian paths from $v$, and part (3) is obtained by combining the first two parts.

**Proposition 8** Let $G$ be $UHCv$. Let $n_i$ = the number of vertices in $V(G) - \{v\}$ of degree $i$ for $i = 2, 3, 4$. Then:
1. $n_2 + n_3 + n_4 = p - 1$,
2. $2n_2 + n_3 - \deg v = p - 1$,
3. $n_2 = n_4 + \deg v$.

The following theorem relating forced edges to penultimate vertices reinforces the belief that penultimate vertices are an important tool in understanding graphs $UHCv$.

**Theorem 9** (Knickerbocker, Lock, Sheard [16]) Let $G$ be $UHCv$. Then $xy$ is a forced edge if and only if $x$ is penultimate to $y$ and $y$ is penultimate to $x$.

The proofs of many of the previous theorems, and most of the results to follow, rely heavily on the concept of the "transform graph" of a graph $UHCv$. This idea originated with Pósa and Thomason (see [22] and [24]) and is one of the most important concepts in this field.

**Definition 10** (Hendry, [9]) Let $G$ be a graph $UHCv$. Define the transform graph of $G$ at $v$, denoted $T(G, v)$ or $T$, as follows:

$V(T) = \{H_x : x \in V(G) \setminus \{v\}\}$,

$H_xH_y \in E(T)$ if and only if $H_y : v...wx...y$ with $w \in N(y)$.

In Definition 10, notice that, since $w \in N(y)$, we have a path to $x$ in the form $H_x : v...wy...x$ by simply reversing the last part of the path to $y$. In this case, we say that $H_y$ can be transformed to $H_x$. It is easily seen that $\deg_T(H_x) = \deg_G(x) - 1$ for all
vertices \( x \neq v \). An example is given in Figure 3 of a graph \( G \) which is UHCv, together with its corresponding transform graph \( T \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Figure 3}
\end{figure}

**Theorem 11** (Hendry, [8, 10]) Let \( G \) be UHCv with transform graph \( T \). Then \( T \) is a forest. Furthermore, the number of components of \( T \) is \((\deg v) / 2\).

In [9], Hendry extends the definition of the transform graph to any graph \( G \) with a distinguished vertex \( v \) as follows: Each vertex of \( T \) corresponds to a hamiltonian path from \( v \). An edge is drawn whenever the hamiltonian path corresponding to one vertex can be transformed to the hamiltonian path corresponding to the other vertex. Hendry has shown, however, that the converse of the preceding theorem is false. Even if we assume that \( G \) is HCv and that \( T(G, v) \) is a forest, \( G \) need not be UHCv.

Transform graphs have not yet been explored in depth for graphs HCv. One of the few results known is due to Hendry (unpublished): If \( G \) is HCv and \( T \) is its transform graph, then \( \text{girth}(T) \geq 6 \) with equality if and only if \( G \) has a hamiltonian path from \( v \) in which the last four vertices induce a \( K_4 \). The following theorem describes more of the structure of the transform graphs for graphs which are UHCv.

**Theorem 12** (KLS, [16]) Let \( G \) be UHCv and let \( T \) be the transform graph of \( G \). Then every component of \( T \) contains a path in the form:

\[ H_{x_1}H_{x_2} \ldots H_uH_u \ldots H_{y_2}H_{y_1} \]

where \( u, w \in N(v), x_1, x_2 \in N(u) \) with \( x_1 \) penultimate to \( u \), and \( y_1, y_2 \in N(w) \) with \( y_1 \) penultimate to \( w \). Furthermore, in each component, these are the only vertices \( H_x \) in which \( x \) is a neighbor of \( v \) or a neighbor of a neighbor of \( v \).

The path described in the above theorem is called the central path of a component of \( T \). The importance of this path is readily apparent from the following observations. If \( x \)
is a neighbor of a neighbor of \( v \) (say \( x \in N(u) \) with \( u \in N(v) \)), then \( H_x \) has the form:
\[
u u \ldots x.
\]
Notice, then, that \( H_x - vu + ux \) is the unique hamiltonian cycle of \( G - v \). Thus we say this cycle is represented by vertices \( H_x \) in \( T \) for every \( x \in N(N(v)) \). Furthermore, if \( u \in N(v) \), then \( H_u + uv \) is a hamiltonian cycle in \( G \). It is easily seen that two vertices corresponding to paths to neighbors of \( v \) are in the same component of \( T \) if and only if they represent the same hamiltonian cycle in \( G \). Thus the components of \( T \) are in a natural one-to-one correspondence with the hamiltonian cycles of \( G \), and the unique hamiltonian cycle of \( G - v \) is represented in every component of \( T \).

Knickerbocker, Lock, and Sheard [16] developed a classification system for all edges in \( G - v \). Let \( ab \) be an edge in \( G - v \). If edge \( ab \) is included in the path \( H_x \), we say \( ab \) covers \( H_x \). The induced subgraph of \( T \) generated by the set of all \( T \)-vertices \( H_x \) covered by \( ab \) will be denoted \( C(ab) \) and will be called the cover of \( ab \) in \( T \). An edge in \( T \) will be called a border of \( C(ab) \) if it is not in \( C(ab) \) but it is incident with a vertex in \( C(ab) \).

**Theorem 13** (KLS, [16]) Assume \( G \) is UHCv with \( \deg v = 2 \), and let \( T \) be the transform graph of \( G \). Then every edge \( ab \) in \( G - v \) falls into one of the following four categories:

1. \( C(ab) \) has no borders. This is the case if and only if \( ab \) is a forced edge.
2. \( C(ab) \) has exactly one border. This is the case if and only if \( a \) is penultimate to \( b \) and \( b \) is not penultimate to \( a \), or vice versa. We will call such an edge an ultimate edge.
3. \( C(ab) \) has two borders and is connected. We will call such an edge a stick.
4. \( C(ab) \) has two borders and is disconnected. We will call such an edge an arrow.

We extend these definitions to any graph \( G \) UHCv (i.e. with \( \deg v > 2 \)) as follows: The definitions of forced edge and ultimate edge are still valid. We call edge \( ab \) a local stick if \( C(ab) \) has two borders, both on the same component of \( T \), and if the intersection of \( C(ab) \) with this component is connected. We call edge \( ab \) a local arrow if \( C(ab) \) has two borders, both on the same component of \( T \), and if the intersection of \( C(ab) \) with this component is disconnected.

Knickerbocker, Lock, and Sheard [16] described the interconnections between forced edges, ultimate edges, (local) sticks, and (local) arrows. The results obtained from this classification of edges in \( G - v \) produce the following three theorems:
Theorem 14 (KLS, [16]) Let $G$ be $UHC_v$ and assume that $\deg v = 2$. Then the unique hamiltonian cycle of $G$ consists exactly of the sets of forced edges, ultimate edges, and sticks, together with the two edges incident with $v$.

Theorem 15 (KLS, [16]) Let $G$ be $UHC_v$, and let $ab$ be an edge in $G$. If $a$ is penultimate to $b$, then edge $ab$ lies on the unique hamiltonian cycle of $G - v$ and on every hamiltonian cycle of $G$.

Theorem 16 (KLS, [16]) A graph $G$ can be uniquely hamiltonian-connected from at most three vertices. Furthermore, if $G$ is $UHC$ from more than one vertex, then all distinguished vertices must have degree two.

Whether a graph can be uniquely hamiltonian-connected from more than one vertex remains an open question. However, the authors do have examples of graphs which are uniquely hamiltonian-connected from one vertex, and hamiltonian-connected from additional vertices. The graph given in Figure 4, for example, is uniquely hamiltonian-connected from $v$ and also hamiltonian-connected from $a$ and $b$. 

Figure 4
One of the broadest remaining open questions involving graphs \( UHCv \) is whether it is possible to completely characterize all such graphs. There are several ways one might approach this question, two of which are discussed here: considering forbidden subgraphs, and finding a class of constructions. We will discuss the progress made on each.

First, we would hope to answer the following question: Which graphs \( H \) may not be subgraphs of any graph which is \( UHCv \)? Hendry [9] has shown that if \( G \) is \( UHCv \), then the following cannot be subgraphs of \( G \): \( K_4 \), \( K_{2,3} \), \( K_1 + P_4 \). Knickerbocker, Lock, and Sheard have extended this list to include \( P_3 \times P_3 \), the "triple bowtie", and the "fish". These six forbidden subgraphs are given in Figure 5. (Graphs a, b, c, e, f in Figure 5 are known to be best possible in that deleting even one edge yields a graph which is not forbidden. It is unknown whether graph d of Figure 5 is best possible.) Ideally, it may be possible to identify a class of graphs such that if \( H \) does not contain any member of this class as a subgraph, then \( H \) may be embedded in a graph which is \( UHCv \).

Second, consider the problem of constructing graphs \( UHCv \). Hendry [8] described four constructions (that is, four methods of obtaining larger graphs \( UHCv \).
from one or more smaller graphs \( \text{UHC}_v \), and a fifth construction is described by him in [9]. Kirchdorfer, Knickerbocker, Lock, and Sheard [14] added to and extended these constructions to obtain a current total of eight constructions. Hendry gave examples in [9] of four graphs \( \text{UHC}_v \) which are not constructible in this manner, and the authors have found more. (Such graphs have been dubbed "rogues"). Still, a constructive characterization of these graphs may well be possible.

4. The Number of Edges in a Graph \( \text{HC}_v \)

Recall that the original motivating question behind the study of graphs \( \text{UHC}_v \) was to find the minimum number of edges in a graph \( \text{HC}_v \). The following lower bound for the number of edges in such a graph is given in [3]:

**Theorem 17** (Chartrand and Nordhaus, [3]) If \( G \) is \( \text{HC}_v \) with \( p \geq 4 \), then the number of edges in \( G \) is greater than or equal to \( \left\lceil \frac{(5p - 1)}{4} \right\rceil \).

The proof of this result is based on the following observation: if \( G \) is \( \text{HC}_v \), then no vertex in \( V(G) - \{v\} \) can be adjacent to two vertices in \( V(G) - \{v\} \) both of degree 2. In [3], Chartrand and Nordhaus gave a class of graphs \( \text{HC}_v \) with approximately \( 3p/2 \) edges. Hendry improved on this when he developed a class of graphs \( \text{HC}_v \) in [8] which have approximately \( 7p/5 \) edges. Finally, in [15], Knickerbocker, Lock, and Sheard gave a class of examples of arbitrarily large order which exactly meet the lower bound given in Theorem 17. Given the "local" nature of the proof of this theorem, it is surprising that the bound is sharp. In addition, it is interesting to note that there are graphs \( \text{HC}_v \) with fewer edges (but more paths!) than graphs \( \text{UHC}_v \).

Figure 6 is an example of a graph which exactly meets this lower bound. Knickerbocker, Lock, and Sheard have shown that if a graph which is \( \text{HC}_v \) has the minimum number of edges as given in the bound of Theorem 14, then the distinguished vertex \( v \) is unique.
Approaching the problem of counting edges from the other direction, Hendry [7] determined the maximum number of edges in a graph which is not hamiltonian-connected from a vertex, as follows:

Theorem 18 (Hendry, [7]) If G is a graph of order $p > 2$ which is not hamiltonian-connected from a vertex, then:
1. $|E(G)| \leq \frac{(p^2 - 3p + 4)}{2}$, and
2. if equality holds in 1, then $G$ is $C_4$, $K_2 + K_3$, or for $p > 3$, $K_2, K_{p-1}$ (i.e. the graph of order $p$ having two blocks: $K_2$ and $K_{p-1}$).

5. The Complexity of HCv and UHCv

One of the most famous computationally difficult problems in graph theory is determining whether a given graph $G$ has a hamiltonian circuit; this problem, which we will call HAM, is NP-complete ([13]). Since determining whether $G$ is HCv or UHCv are closely related problems, it is natural to ask whether these problems are also hard in this sense. Because of the many structural restrictions (odd number of vertices, $(3p - 3)/2$ edges, all vertices of degree 2, 3, or 4, etc.), the complexity of deciding
whether $G$ is $\text{UHC}_v$ is of particular interest. One might hope that all these restrictions might make the time complexity of this problem polynomial. On the other hand, very restricted forms of the hamiltonian cycle problem remain NP-complete (e.g., restricted to planar bipartite graphs with all vertex degrees either 2 or 3 ([12])). In this section, we discuss what is known about the complexity of $\text{HC}_v$ and $\text{UHC}_v$.

For a review of the basic ideas and terminology of complexity theory, we refer the reader to [6], [17], [20], or [12]. Throughout this section, we will follow the notation and terminology of [12].

In Table 1, we list three decision problems that are relevant to our discussion:

- **$\text{HC}_v$**
  - **Instance:** A graph $G$ and distinguished vertex $v$.
  - **Question:** Is $G$ hamiltonian-connected from $v$?

- **$\text{2HC}_v$**
  - **Instance:** A graph $G$ and distinguished vertex $v$.
  - **Question:** Is $G$ hamiltonian-connected from $v$ in more than one way, i.e., is there a hamiltonian path from $v$ to each other vertex $x$, and is there at least one vertex $x$ for which there is more than one such path?

- **$\text{UHC}_v$**
  - **Instance:** A graph $G$ and distinguished vertex $v$.
  - **Question:** Is $G$ uniquely hamiltonian-connected from $v$?

Table 1

Using a transformation from HAM, Dean [5] has determined the complexity of the first two problems:

**Theorem 19** (Dean, [5]) $\text{HC}_v$ and $\text{2HC}_v$ are NP-complete.

The complexity of $\text{UHC}_v$ is apparently much more difficult to determine. To begin with, it is not at all obvious that $\text{UHC}_v$ is even an element of NP. To verify that a particular graph $G$ is uniquely hamiltonian-connected from vertex $v$, one would not only have to demonstrate that hamiltonian paths exist from $v$ to each other vertex $x$, but also that each such path was unique. On the other hand, if $\text{UHC}_v$ graphs could be characterized in terms of a list of structural restrictions that could be checked in polynomial time, then $\text{UHC}_v$ would actually be an element of P.

There is a class of decision problems for which $\text{UHC}_v$ is a natural member; it is the class $\text{DP}$, which was defined by Papadimitriou and Yannakakis in 1984 ([21]). $\text{DP}$ is defined to be the class consisting of those decision problems which are the intersection of a problem in NP with one in coNP. In other words, each problem $Q$ in $\text{DP}$ is defined in terms of two problems $Q_1$ and $Q_2$ in NP with the same set of instances, such that the answer for $Q$ is yes if and only if the answer for $Q_1$ is yes and the answer for $Q_2$ is no.
Note that NP and coNP are subclasses of DP, since any problem Q is the intersection of itself with the trivial problem having the same instances and a question whose answer is always yes. It is easy to see that UHC_\text{v} is an element of DP, since UHC_\text{v} = HC_\text{v} \cap 2HC_\text{v}.

The problem UHAM (Given G, does it have a unique hamiltonian cycle?) is another natural member of DP. UHAM is NP-hard and has the additional property that if it is in NP, then NP = coNP. The proof follows from a transformation used in [12] which shows that 2HAM is NP-complete. However, an argument using the analogous transformation for 2HC_\text{v} fails to show that UHC_\text{v} has these properties.

A problem Q in DP is DP-complete if any problem in DP can be polynomially transformed to Q. Note that any DP-complete problem Q has the properties described above. It is not known whether UHAM is DP-complete. Indeed, Blass and Gurevich [1] have shown that it may be impossible to answer this question using standard techniques, by constructing two oracles for which the relative DP classes properly contain NP \cup coNP, but such that the closely related problem USAT (Given a boolean formula, does it have a unique satisfying truth assignment?) is DP-complete relative to one oracle and not DP-complete relative to the other. For more details on the complexity of unique problems and the class DP, see [11], [12], or [21].

6. Conclusion

In this section we discuss open problems and current research directions, some of which have already been mentioned in passing. The concentration of previous research in the more restrictive area of graphs UHC_\text{v} leads to a larger collection of specific questions there than for the more general case of HC_\text{v}, but this should not be taken as a measure of relative importance. Indeed, in the realm of graphs HC_\text{v}, we suspect that the most interesting questions have not yet been formulated.

The most promising area for research on graphs HC_\text{v} appears at present to be an exploration of transform graphs. The hope is that a fuller understanding of transform graphs will provide the kind of insights into the structure of graphs HC_\text{v} which has already been provided for graphs UHC_\text{v}. Currently, very little is known. It is easy to generate natural conjectures and questions; for example, Hendry [private communication] has posed the following:

*Problem* Determine necessary and sufficient conditions for a graph HC_\text{v} to have a connected transform graph.

*Problem* Determine necessary and sufficient conditions for two graphs HC from distinguished vertices to have isomorphic transform graphs.
If the class of graphs $HC^v$ appears too large to tackle all at once, a reasonable starting point is the class of graphs minimally Hamiltonian-connected from $v$: $G$ is minimally $HC^v$ if it is $HC^v$, but the deletion of any edge yields a graph which is not $HC^v$. Such graphs may have nice structural properties, similar to those of graphs $UHC^v$ (which are a subclass). Any results obtained may then lift in some fashion to the broader class of $HC^v$, since clearly any graph $HC^v$ contains a minimally $HC^v$ subgraph. We add the following:

**Problem** Determine $\max \{ |E(G)| : |V(G)| = p \text{ and } G \text{ is minimally } HC^v \}$.

As suggested earlier, perhaps the outstanding open problem concerning graphs $UHC^v$ is to provide a constructive characterization: a small list of graphs with which to begin and a few methods of making larger graphs out of smaller ones, whereby the class of all graphs $UHC^v$ can be generated. The evidence is ambiguous as to whether such a characterization is possible. On one hand, at the largest orders for which a systematic study has been carried out ($p = 11, 13,$ and $15$), rogues appear which bear little resemblance to any smaller graphs $UHC^v$. On the other hand, the huge majority of graphs $UHC^v$ of each of these orders are easily constructible using known constructions. The issue may well depend on whether more regularity or less emerges at higher orders.

While the problem of characterizing graphs $UHC^v$ remains mysterious, a characterization of their transform graphs may be within reach. In particular, the structure of the transform tree where $\deg v = 2$ is already partially understood.

We turn now to forbidden subgraphs for graphs $UHC^v$. Some organized understanding of this topic is sorely needed, if it is to be more than just a collection of amusing but largely unrelated puzzles. The ideal would be to give a simple characterization of a class $\Phi$ of graphs (the "atomic forbidden subgraphs") such that $H$ is a subgraph of a graph $UHC^v$ if and only if $H$ has no member of $\Phi$ as a subgraph. What such a $\Phi$ might look like, or exactly how simple its description might be, is currently most unclear. At the moment, the proofs for the few known forbidden subgraphs appear to be very much ad hoc, so nothing like an organized pattern has yet emerged.

The problem of showing that a graph can be $UHC^v$ from at most one vertex remains stubbornly open, although Theorem 16 and some more technical work of the authors have greatly restricted the terrain in which a counterexample might be found. We continue to believe that the distinguished vertex is unique. In any case counterexamples will not be easily found; we have shown that a graph which is $UHC$ from three vertices...
(the maximum possible under Theorem 16) must have order at least 27 and size at least 39, which makes even a theory-driven search virtually impossible.

Finally, there are several open questions concerning the computational complexity of UHCv: Is UHCv in NP or does it have the property, like UHAM, that if it is in NP, then NP = coNP? If the answer to the latter question is yes, then UHCv have the stronger property of being \( \text{D}^\text{P} \)-complete?

While a great deal of progress has been made in understanding graphs hamiltonian-connected from a vertex and graphs uniquely hamiltonian-connected from a vertex, much remains to be done. With open questions ranging from hands-on construction techniques to computational complexity to grand classification schemes, the field of graphs hamiltonian-connected from a vertex is ripe for further research.

REFERENCES


