Final Review Answers

1. Evaluate each of the following limits:

- \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{3-x} = \lim_{x \to 3} \frac{(x+1)(x-3)}{(3-x)} = \lim_{x \to 3} -(x+1) = -4. \)

- \( \lim_{t \to \infty} \frac{t^7 - 5t^6 + 71.6}{6t^6 - 18.3t^2 + 71.6} = \lim_{t \to \infty} \frac{1/t^7 - 5/6t^5 + 71.6/t^4}{6 - 18.3/t^4 + 71.6/t^4} = 0/6 = 0. \) This could also be done by L'Hospital's Rule.

- \( \lim_{h \to 0} \frac{\sin(\pi/2+h)-\sin(\pi/2)}{h} = \sin' (\pi/2) = \cos(\pi/2) = 0. \)

2. Find the derivatives of the following functions:

- \( f(x) = x^2 + 5x - 3. \) \( f'(x) = 2x + 5. \)

- \( g(x) = x^{(2/3)} \tan(x). \) \( g'(x) = (2/3) x^{(-1/3)} \tan(x) + x^{(2/3)} \sec^2(x). \)

- \( h(x) = \frac{(t^2-1)}{(t-1)^2} \) \( h'(x) = \frac{1(t^2-1)-2t(t+1)}{(t-1)^2} = \frac{-t^2-2t-1}{(t-1)^2(t+1)} = \frac{-t-1}{t^2(t+1)} = \frac{-1}{(t-1)^2}. \) An easier way to do this problem is to simplify first: \( h(x) = \frac{1}{t-1} = (t-1)^{-1}, \) so \( h'(x) = -(t-1)^{-2}. \)

- \( m(t) = (t^3 + \cos(t))^{12}. \) \( m'(t) = 12(t^3 + \cos(t))^{11} \cdot (3t^2 - \sin(t)). \)

- \( n(x) = \arcsin(\tan(e^x)). \) \( n'(x) = \frac{1}{\sqrt{1-(\tan(e^x))^2}} \cdot (\sec^2(e^x) \cdot (e^x)). \)

3. Find \( y' = \frac{dy}{dx} \) by implicit differentiation, where \( y \) is a function of \( x \) defined by the relation \( x y^3 + \sin(y) = 0. \)

We take the derivative on both sides of the relation with respect to \( x, \) viewing \( y \) as a function of \( x: \)

\[ y^3 + 3x y^2 y' + \cos(y) y' = 0. \]

Solving for \( y', \) we obtain

\[ y' = \frac{-y^3}{3x y^2 + \cos(y)}. \]
4. Evaluate the following using the FTC:

- \[ \int_1^3 \frac{x+7}{x} \, dx = \int_1^3 \left(1 + \frac{7}{x}\right) \, dx = (x + 7 \ln(x)) \bigg|_1^3 = (3 + 7 \ln(3)) - (1 + 7 \ln(1)) = 2 + 7 \ln(3). \]

- \[ \int_0^7 \cos(x) \, dx = \sin(x) \bigg|_0^7 = \sin(7) - \sin(0) = 0 - 0 = 0. \]

The graph of \( y = f(x) = \cos(x) \) on \([0, \pi]\) makes this result clear: one has two congruent regions whose signed areas "cancel out."

\[
\begin{align*}
y &= \cos(x) \text{ on } [0, \pi] \\
\end{align*}
\]

- \[ \int_0^1 \frac{1}{1+x^2} \, dx = \arctan(x) \bigg|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4} \approx .785. \]

A graph shows this value is reasonable (note the rectangle has area 1).

\[
\begin{align*}
y &= 1/(1+x^2) \\
\end{align*}
\]

5. Let \( f \) be the function whose graph is shown, and let \( F \) be defined by \( F(x) = \int_0^x f(t) \, dt \).

Answer the following questions:

\[
\begin{align*}
y &= f(x) \\
\end{align*}
\]
a) Estimate $F(5)$ and $F(1)$:

$$F(5) = \int_3^5 f(x) \, dx = \text{signed area of "triangle" on } [3, 5], \text{ which is } -(1/2) b \cdot h = -(1/2)(2)(4/3) = -4/3.$$  
$$F(1) = \int_1^3 f(t) \, dt = -\int_1^3 f(t) \, dt = -4/3, \text{ since the triangle on } [1, 3] \text{ is congruent to the triangle on } [3, 5].$$

b) Where is $F$ increasing? Explain your answer.

$F$ is increasing where $F' = f$ is positive, which is on the interval $(-3, 3)$. (Why is $F' = f$? By the first part of the Fundamental Theorem of Calculus!)

c) At which value(s) of $x$ does $F$ have local minima? Explain.

A local minimum of $F$ occurs at a critical point such that $F'$ is negative to the left and positive to the right. Since $F' = f$, the critical points are -3 and +3, the values of $x$ where $F'(x) = f(x) = 0$. Of these, -3 has $F'$ negative to the left and positive to the right, so $x = -3$ is a local minimum of $F$.

d) Where is $F$ concave down? Explain.

$F$ is concave down where $F' = f$ is decreasing, which is on the interval $(0, 5)$ -- or $(0, \infty)$, if trends continue as shown.

6. Referring to the function $f$ whose graph is given in the previous problem, sketch the graph of $g$, where $g(x) = f(x + 2) + 1$. Also sketch the graph of $F$ that was discussed in the previous problem.

The graph of $g$ is obtained by translating the graph of $f$ to the left by 2 units and vertically upward by 1 unit. We obtain:

Recall that $F$ is the antiderivative of $f$ that passes through the point $(3, 0)$. It has a local minimum at -3 and a local maximum at +3. Here is the graph:
7. Use the definition of the derivative (limit process) to compute \( h'(x) \), where \( h(x) = \frac{1}{x} \). Then find the equation of the line tangent to the graph at the point \((2, 1/2)\).

By definition, \( h'(x) = \lim_{z \to x} \frac{h(z) - h(x)}{z - x} = \lim_{z \to x} \frac{\frac{1}{z} - \frac{1}{x}}{z - x} = \lim_{z \to x} \frac{x - z}{xz(z - x)} = \lim_{z \to x} \frac{-1}{xz} = -\frac{1}{x^2} \). So, the slope of the curve at \( x = 2 \) is \( h'(2) = -1/4 \), and the equation of the tangent line is \( y - (1/2) = (-1/4)(x - 2) \), or \( y = (-1/4)x + 1 \).

\[
\text{Plot}[[1/x, (-1/4)x + 1], (x, -2, 4)]
\]
8. Sketch a graph of a single function $f$ with the following properties:

- $f(0) = 1$, $f(1) = 4$, $f(-2) = 0$;
- $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$;
- $f'(-2) = f'(3) = 0$, and $f'(1)$ is not defined;
- $f'(x) > 0$ for $x < -2$, $-2 < x < 1$, and $x > 3$;
- $f'(x) < 0$ for $1 < x < 3$.

9. A cannonball is fired directly upward, starting from ground level (height = 0), at time $t = 0$ seconds. Let $h(t)$ represent the cannonball’s height above the ground level (in feet) at time $t$ seconds, $v(t) = 192 - 32t$ the cannonball’s vertical velocity (in feet per second) at time $t$, and $a(t)$ the vertical acceleration at time $t$ seconds.

- **a)** Calculate $\int_{1}^{4} v(t) \, dt$. What does this answer tell you about the cannonball?
- **b)** Showing all work, find a formula for $h(t)$.
- **c)** Showing all work, find a formula for $a(t)$.
- **d)** At what time is the cannonball at the highest point? How high is it at that time?

- **a)** $\int_{1}^{4} v(t) \, dt = \int_{1}^{4} (192 - 32t) \, dt = (192t - 16t^2)|_{1}^{4} = (192\cdot4 - 16\cdot4^2) - (192 - 16) = 512 - 176 = 336$. This is the displacement of the cannonball on the interval [0, 4]; that is, it is the net change in height of the cannonball on this interval. This is because $h(t)$ is an antiderivative of $v(t)$, so the integral equals $h(4) - h(1) = \text{final height} - \text{initial height}$.

- **b)** As just observed, $h(t)$ is an antiderivative of $v(t)$, so $h(t) = 192t - 16t^2 + C$. Since $h(0) = 0$, we see $C = 0$; therefore, $h(t) = 192t - 16t^2$.

- **c)** Since $a(t) = v'(t)$, we get that $a(t) = (192 - 32t)' = -32$ (ft/sec)/sec.

- **d)** The cannonball is at its highest point when it is instantaneously motionless: Solve $v(t) = 0$ to find the time of maximum height. $192 - 32t = 0 \Rightarrow t = 192/32 = 6$. So the maximum height is attained at time $t = 6$ sec, and $h(6) = 192\cdot6 - 16\cdot6^2 = 576$ ft is the maximum height attained.
10. Use the average of the left and right sums with four subintervals to approximate the value of \( \int_{1}^{3} \frac{6}{x} \, dx \).

If we divide the interval \([1, 3]\) into four equal subintervals, their common length is \( \Delta x = \frac{3-1}{4} = \frac{1}{2} \). The subinterval endpoints are \( 1 = 2/2, \ 3/2, \ 4/2, \ 5/2, \) and \( 6/2 = 3 \). The left sum is therefore \((1/2) \cdot (f(1)+f(3/2)+f(2)+f(5/2))\), which is equal to 7.7:

\[
\begin{align*}
f[x_] &= 6/x; \\
LeftSum[f, 1, 3, 4]
\end{align*}
\]

The sum = 7.7

The right sum is \((1/2) \cdot (f(3/2) + f(2) + f(5/2) + f(3))\), which is equal to 5.7:

\[
\begin{align*}
RightSum[f, 1, 3, 4]
\end{align*}
\]

The sum = 5.7

The average of the left and right sums is \((7.7+5.7)/2 = 6.7\). This should be pretty close to the exact area under the curve, and also pretty close to the Midpoint approximation:
MidpointSum[f, 1, 3, 4]

The sum = 6.53853

11. Evaluate the following using geometry:

- \( \int_{-1}^{2} |x| \, dx = \int_{-2}^{1} |x| \, dx \)

From the graph, we see that \( \int_{-2}^{1} |x| \, dx = \) (area of left triangle) + (area of right triangle) = \( 1/2(2)(2) + 1/2(1)(1) = 2.5 \). Therefore, \( \int_{-1}^{2} |x| \, dx = -2.5 \).
\[ \int_{-1}^{2} (2 - 3x) \, dx \]

\[
\text{Plot}[2 - 3x, \{x, -1, 2\}, \text{PlotLabel} \rightarrow \text{"y = 2 - 3x"}]
\]

The integral in this case is (area of left triangle) - (area of right triangle). The x-intercept of the line is \( x = 2/3 \). So, the base of the left triangle is \( 1 + 2/3 \) and the base of the right triangle is \( 2 - 2/3 = 4/3 \). The height of the left triangle is \( 2 - 3(-1) = 5 \) and the height of the right triangle is \( -(2 - 3(2)) = -(-4) = 4 \). So the integral is given by

\[
\frac{1}{2} \left( \frac{1}{2} + \frac{2}{3} \right) (5) - \left( \frac{1}{2} \right) \left( \frac{4}{3} \right) (4)
\]

\[
\frac{3}{2}
\]

12. Find the local maxima and minima of the function \( f \), given by \( f(x) = 3x^3 - 9x \), and the global maximum and minimum (outputs) of \( f \) on the interval \([-2,3]\).

The domain of the function is all reals, and \( f \) is differentiable everywhere, so the only critical points are those of the form \( f'(x) = 0 \). The derivative is \( f'(x) = 9x^2 - 9 = 9(x^2 - 1) \), which is 0 at \( x = \pm 1 \). We see that the derivative is positive for \( x < -1 \) and \( x > 1 \), and negative on \((-1, 1)\), so \( f \) is increasing on \((-\infty, -1) \cup (1, \infty)\) and decreasing on \((-1, 1)\), so \( x = -1 \) is a local max and \( x = 1 \) is a local min. On the interval \([-2, 3]\), the global max and the global min can occur either at a critical point or an endpoint.

\[
f[x_] := 3x^3 - 9x
\]

\[
\{f[-2], f[-1], f[1], f[3]\}
\]

\[
\{-6, 6, -6, 54\}
\]

We see that the global min output is -6, which occurs at \( x = -2 \) and \( x = 1 \), and the global max output is 54, which occurs at \( x = 3 \). Here is the plot:
13. You wish to mail a cylindrical package whose combined length and girth (the circumference of a cross section perpendicular to the length) is 84 inches. What are the length and girth of the cylindrical tube with the largest volume that you can mail?

Let \( r \) be the radius of the package, and \( l \) be its length. The girth is then \( 2\pi r \), and we want as large a volume as possible, so we choose a package for which length + girth = \( l + 2\pi r = 84 \), so \( r = \frac{84-l}{2\pi} \). The volume of the package is (base area) \( \times \) (length), which is \( \pi r^2 l = \pi \left( \frac{84-l}{2\pi} \right)^2 l = \left( \frac{1}{4\pi} \right) (84^2 l - 168 l^2 + l^3) \). The domain is \([0, 84] \), clearly. Taking the derivative of the volume and setting it equal to 0, we obtain \( 84^2 - 336 l + 3 l^2 = 0 \), or \( 84 \cdot 28 - 112 l + l^2 = 0 \), or \((l - 28)(l - 84) = 0 \), so the critical points are \( l = 28 \) and \( l = 84 \). From the factored form of the volume function, we see that the volume is 0 when \( l = 84 \) (and \( l = 0 \)), so the max volume occurs when \( l = 28 \) and (therefore) the girth = 56. So the maximum volume occurs when the girth is twice the length.

14. Compute the definite integral of the function \( h \), given by \( h(x) = \frac{1}{x^2} \), on the interval \([1/2, 5/2] \).

\[
\int_{1/2}^{5/2} \frac{1}{x^2} \, dx = \left[ \frac{-1}{x} \right]_{1/2}^{5/2} = (-2/5) - (-2/1) = 2 - 2/5 = 8/5. \]  
(We used the Fundamental Theorem of Calculus.) We can check with Mathematica's integration function:

\[
\int_{1/2}^{5/2} \frac{1}{x^2} \, dx
\]
15. Use linear approximation to estimate $\sqrt{9.01}$. [Hint: find the best linear approximation to $f(x) = \sqrt{x}$ at $x = 9$.]

The best linear approximation (or linearization) of $f$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. For us, $f(x) = \sqrt{x}$, so $f(9) = 3$ and $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. Therefore, $L(x) = 3 + \frac{1}{6}(x - 9)$, so $\sqrt{9.01} = f(9.01) \approx L(9.01) = 3 + \frac{1}{6}(9.01 - 9) = 3 + 0.01/6 = 3.0016666...$ Here is the exact value computed by *Mathematica* to 10 decimal places:

$$\sqrt{9.01} \approx 3.001666204$$

Here is a picture of the function and its tangent line at $x = 9$:

```
Plot[{Sqrt[x], 3 + (1/6) (x - 9)}, {x, 1, 16}, PlotRange -> {.5, 4.5}]
```

16. Find the point that the Mean Value Theorem guarantees will exist for the function $f(x) = \sqrt{x}$ on the interval $[1, 9]$.

The Mean Value Theorem requires that the function be continuous on the closed interval and differentiable on the open interval; this is clearly true for this function and interval. The point $c$ we seek satisfies $f'(c) = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$. So $f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{4}$, so the solution is $c = 4$. The graph illustrates the geometric content: the slope of the chord equals the slope of the tangent at $(c, f(c))$. 
17. Sketch the graph of \( y = f(x) = \frac{x^2}{x^2 + 3} \) on the axes provided. Indicate intervals on which the function is increasing and decreasing, concave up and concave down, and label any critical points and points of inflection.

In[8]:= \[ f[x_] := x^2 / (x^2 + 3) \]

Note: we took the derivatives by hand in the review session.

In[10]:= Simplify[f'[x]]

Out[10]= \[ \frac{6 x}{(3 + x^2)^2} \]

Taking the derivative and simplifying, we obtain the expression above, which is equal to 0 at \( x = 0 \). The derivative has negative values to the left of 0, and positive values to the right, so the function \( f \) decreases on \((-\infty, 0)\) and increases on \((0, \infty)\). This makes \( x = 0 \) a local and global minimum of the function; there are no local maxima.

The second derivative is given by

In[11]:= Simplify[f''[x]]

Out[11]= \[ -\frac{18 (-1 + x^2)}{(3 + x^2)^3} \]

Factoring the numerator, it is easily seen that the second derivative is 0 at -1 and at 1. Furthermore, \( f'' \) is negative to the left of -1 and to the right of 1, so the function \( f \) is concave down on those intervals, and \( f'' \) is positive on \((-1, 1)\), so \( f \) is concave up on that interval. Therefore, +1 and -1 are the points of inflection of \( f \). The plot confirms all of these observations:

In[12]:= Plot[f[x], {x, -3, 3}, AspectRatio -> Automatic]

Out[12]=